Volume 67B
Number 1
Jan.–Mar. 1963

Mathematics and Mathematical Physics

U.S. DEPARTMENT OF COMMERCE

SECTION B

JOURNAL OF RESEARCH

NATIONAL BUREAU OF STANDARDS

Journal of Research

of the

National Bureau of Standards

B. Mathematics and Mathematical Physics

JAN.–MAR. • 1963

VOLUME • 67B

NUMBER • 1

Editor: Chester H. Page

Associate Editors: William H. Pell, John J. Sopka

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1963

JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles on topics closely related to the Bureau's scientific and technical program.

The Journal is published in four separate sections as follows:

A. Physics and Chemistry

Contains papers of interest primarily to scientists working in these fields. Issued six times a year. Annual subscription: domestic, $4.00; foreign, $4.75.

B. Mathematics and Mathematical Physics

Presents studies and compilations designed mainly for the mathematician and the theoretical physicist. Issued quarterly. Annual subscription: domestic, $2.25; foreign, $2.75.

C. Engineering and Instrumentation

Reports research and development results of interest chiefly to the engineer and the applied scientist. Issued quarterly. Annual subscription: domestic, $2.25; foreign, $2.75.

D. Radio Propagation

Reports work in radio propagation, upper atmospheric physics, and communications. Issued six times a year. Annual subscription: domestic, $4.00; foreign, $4.75.

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. The Bureau also serves as the Federal technical research center in a number of specialized fields. The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work.

Washington, D.C.

Electricity. Resistance and Reactance. Electrochemistry. Electrical Instruments. Magnetic Measurements. Dielectrics. High Voltage.

Metrology. Photometry and Colorimetry. Refractometry. Photographic Research. Length. Engineering Metrology. Mass and Scale. Volumetry and Densimetry.

Heat. Temperature Physics. Heat Measurements. Cryogenic Physics. Equation of State. Statistical Physics.

Radiation Physics. X-Ray. Radioactivity. Radiation Theory. High Energy Radiation. Radiological Equipment. Nucleonic Instrumentation. Neutron Physics.

Analytical and Inorganic Chemistry. Pure Substances. Spectrochemistry. Solution Chemistry. Standard Reference Materials. Applied Analytical Research. Crystal Chemistry.

Mechanics. Sound. Pressure and Vacuum. Fluid Mechanics. Engineering Mechanics. Rheology. Combustion Controls.

Polymers. Macromolecules: Synthesis and Structure. Polymer Chemistry. Polymer Physics. Polymer Characterization. Polymer Evaluation and Testing. Applied Polymer Standards and Research. Dental Research.

Metallurgy. Engineering Metallurgy. Microscopy and Diffraction. Metal Reactions. Metal Physics. Electrolysis and Metal Deposition.

Inorganic Solids. Engineering Ceramics. Glass. Solid State Chemistry. Crystal Growth. Physical Properties. Crystallography.

Building Research. Structural Engineering. Fire Research. Mechanical Systems. Organic Building Materials. Codes and Safety Standards. Heat Transfer. Inorganic Building Materials. Metallic Building Materials.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics. Operations Research.

Data Processing Systems. Components and Techniques. Computer Technology. Measurements Automation. Engineering Applications. Systems Analysis.

Atomic Physics. Spectroscopy. Infrared Spectroscopy. Far Ultraviolet Physics. Solid State Physics. Electron Physics. Atomic Physics. Plasma Spectroscopy.

Instrumentation. Engineering Electronics. Electron Devices. Electronic Instrumentation. Mechanical Instruments. Basic Instrumentation.

Physical Chemistry. Thermochemistry. Surface Chemistry. Organic Chemistry. Molecular Spectroscopy. Elementary Processes. Mass Spectrometry. Photochemistry and Radiation Chemistry.

Office of Weights and Measures.

Boulder, Colo.

Cryogenic Engineering Laboratory. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Cryogenic Technical Services.

Central Radio Propagation Laboratory

Ionosphere Research and Propagation. Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services. Vertical Soundings Research.

Radio Propagation Engineering. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.

Radio Systems. Applied Electromagnetic Theory. High Frequency and Very High Frequency Research. Frequency Utilization. Modulation Research. Antenna Research. Radiodetermination.

Upper Atmosphere and Space Physics. Upper Atmosphere and Plasma Physics. High Latitude Ionosphere Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

Radio Standards Laboratory

Radio Physics. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time-Interval Standards. Radio Plasma. Millimeter-Wave Research.

Circuit Standards. High Frequency Electrical Standards. High Frequency Calibration Services. High Frequency Impedance Standards. Microwave Calibration Services. Microwave Circuit Standards. Low Frequency Calibration Services.

U.S. DEPARTMENT OF COMMERCE
Luther H. Hodges, Secretary

NATIONAL BUREAU OF STANDARDS
A. V. Astin, Director

Order all publications from the Superintendent of Documents
U.S. Government Printing Office, Washington 25, D.C.

Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 19, 1961).

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B No. 1, January–March 1963

Evaluation of a Generalized Elliptic-Type Integral[1]

Leo F. Epstein[2] and J. H. Hubbell

(October 23, 1962)

Integrals of the form

$$\Omega_j(k) \equiv \int_0^{\pi} [1-k^2 \cos\phi]^{-(j+\frac{1}{2})} d\phi$$

where $0 \leq k < 1$, and j is a positive integer, occur in a radiation field problem. Expressions for $\Omega_0(k)$ and $\Omega_1(k)$ have been derived in terms of complete elliptic integrals of the first and second kinds. Using these values, and the recursion formula

$$(2j-1)(1-k^4)\Omega_j(k) = 4(j-1)\Omega_{j-1}(k) - (2j-3)\Omega_{j-2}(k)$$

$\Omega_j(k)$ can be found for all values of j and k. A number of useful series expansions and other relations are given for $\Omega_j(k)$, and tables are included for $0 \leq j(1) \leq 9$ and $0 \leq k^2(0.01) \leq 0.99$.

1. Introduction

In calculating the radiation field off-axis from a uniform circular disk radiating according to an arbitrary angular distribution law, it has been convenient [1][3] to use the method of Legendre polynomial expansion of the source-detector geometry [2]. Evaluation of the coefficients in these expansions leads to a family of integrals

$$\Omega_j(k) \equiv \int_0^{\pi} [1-k^2 \cos\phi]^{-(j+\frac{1}{2})} d\phi \tag{1}$$

where $0 \leq k < 1$. This function has a singularity at $k=1$ which corresponds, in the problem described above, to the detector at the edge of the disk.

The integral (1) is closely related to the complete elliptic integrals of the first and second kind. Thus it can be shown readily that [3, eq 291.00]

$$\Omega_0(k) = (\sqrt{2}\kappa/k) K(\kappa) \tag{2}$$

where

$$\kappa^2 = 2k^2/(1+k^2) \tag{3}$$

and $K(\kappa)$ is the complete elliptic integral of the first kind [3, eq 110.06]

$$K(\kappa) \equiv \int_0^{\pi/2} (1-\kappa^2 \sin^2\phi)^{-\frac{1}{2}} d\phi. \tag{4}$$

Similarly, $\Omega_1(k)$ can be expressed [3, eq 291.01] as

$$\Omega_1(k) = (\sqrt{2}\kappa/k)[1/(1-k^2)] E(\kappa) \tag{5}$$

where [3, eq 110.07]

$$E(\kappa) \equiv \int_0^{\pi/2} (1-\kappa^2 \sin^2\phi)^{+\frac{1}{2}} d\phi. \tag{6}$$

---

[1] Work supported jointly by the Office of Civil Defense (DOD), the Navy Bureau of Yards and Docks, and the Defense Atomic Support Agency.
[2] General Electric Company, Atomic Power Equipment Department, Vallecitos Atomic Laboratory, Pleasanton, Calif.
[3] Figures in brackets indicate the literature references at the end of this paper.

$\Omega_j(k)$ for $j \geq 2$ can also be developed in terms of these functions. Thus, using the integration by parts formula given by Peirce-Foster [4, eq 317], it follows at once that

$$(2j-1)(1-k^4)\Omega_j(k) = 4(j-1)\Omega_{j-1}(k) - (2j-3)\Omega_{j-2}(k). \tag{7}$$

This recursion formula is exact, and by using it and values of $K(\kappa)$ and $E(\kappa)$, $\Omega_j(k)$ can be obtained for all values of j. For example, it is found by inserting (2) and (5) in (7) that

$$\Omega_2(k) = (\sqrt{2}\kappa/k)[(1/3)(1-k^4)][4E(\kappa)/(1-k^2) - K(\kappa)] \tag{8}$$

and

$$\Omega_3(k) = (\sqrt{2}\kappa/k)[(1/15)(1-k^4)^2][+(23+9k^4)E(\kappa)/(1-k^2) - 8K(\kappa)]. \tag{9}$$

It is clear that continuing in the same way $\Omega_j(k)$ can be obtained for all values of j in terms of K and E. Extensive tables of these complete elliptic integrals exist [3, 5, and 6] in addition to excellent polynomial approximations suitable for use on a high speed computer, due to Hastings [7], and a large number of exact and approximate analytical formulations [6, 8].

For values of k^2 approaching unity, the expansions (see, e.g., [5, eqs 773.3 and 774.3])

$$K(\kappa) = \ln\,(4/\kappa') + (1/4)\{\ln\,(4/\kappa') - 1\}\kappa'^2 + \ldots \tag{10}$$

and

$$E(\kappa) = 1 + (1/2)\{\ln\,(4/\kappa') - 1/2\}\kappa'^2 + \ldots, \tag{11}$$

where $\kappa' = \sqrt{1-\kappa^2}$, can be used to compute $\Omega_0(k)$ and $\Omega_1(k)$ to high precision.

This technique for finding $\Omega_j(k)$ has several serious disadvantages, however; e.g., (a) to find Ω_j from (7), particularly for large values of j, the precision of the Ω_j thus computed may be poor due to the buildup of round-off errors; (b) in order to find Ω_j for a given k, it is first required that Ω_0, Ω_1, Ω_2, . . . Ω_{j-1} be known. If only a few values of $\Omega_j(k)$ are required, this method may be painfully long and complicated. For these and similar reasons, alternate methods for calculating $\Omega_j(k)$ are considered below.

As will also appear in the sections which follow, the quantities $\Omega_j(k)$ are related to an interesting collection of transcendental functions. In addition to the elliptic integrals above, it will be shown that $\Omega_j(k)$ is expressible in terms of Bessel functions with a purely imaginary argument, and in terms of error functions. The analysis of these functions thus involves a fascinating excursion into a number of curious bypaths in higher mathematical analysis, in addition to the practical application to the physical problem mentioned in the opening sentences above.

2. Series and Other Expansions for $\Omega_j(k)$

2.1. Simple Power Series $j \geq 0$

For small values of k, $\Omega_j(k)$ can be derived by a binomial theorem expansion of $[1-k^2\cos\phi]^{-(j+\frac{1}{2})}$ in powers of $(k^2\cos\phi)$, i.e.,

$$\Omega_j(k) = \sum_{i=0}^{\infty} \binom{-j-\frac{1}{2}}{i} (-k^2)^i \int_0^{\pi} \cos^i\phi\, d\phi \tag{1}$$

where

$$\binom{n}{i} \equiv n!/i!(n-i)! \tag{2}$$

is the familiar notation for the binomial coefficient[4] [9, par. 1.51], [10]. The integral in (1) can be shown to be [4, eq 498]

[4] As indicated in [9] and [10], binomial coefficients of the form $\binom{n/2}{m}$ are easily generated using the recursions:

$$\binom{n/2}{0} = 1;\ \binom{n/2}{m} = \binom{n/2}{m-1}(n-2m+2)/(2m) = \binom{n/2-1}{m-1} + \binom{n/2-1}{m}.$$

$$\int_0^{\pi} \cos^i\phi\, d\phi = [1+(-1)^i](\sqrt{\pi}/2)\, \frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)} \tag{3}$$

and so vanishes when i is odd. Using this relation and the well known properties of the binomial coefficients, it follows that

$$\Omega_j(k) = \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} \tag{4}$$

where

$$\alpha_m(j) = \frac{\pi}{(64)^m}\, \frac{j!}{(2j)!}\, \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m!}\right)^2. \tag{5}$$

Equation (4) may easily be shown to be convergent for all values of j and for $k \leq 1$. For k close to unity, however, it tends to converge rather slowly; and for this reason, this power series formulation is most useful for $k << 1$.

2.2. Difference Sum, $j \geq 0$

The behavior of (4) may be improved by using a technique previously developed for improving the convergence of series [8]. By this method, the series (4) is written in the identically equivalent form

$$\Omega_j(k) = \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} = \sum_{m=0}^{\infty} \alpha_m^*(j) k^{4m} + \sum_{m=0}^{\infty} [\alpha_m(j) - \alpha_m^*(j)] k^{4m} \tag{6}$$

where $\alpha_m^*(j)$ is defined, first, by the condition

$$\lim_{m \to \infty} [\alpha_m^*(j)/\alpha_m(j)] = 1 \tag{7}$$

that is, α_m^* is asymptotically equivalent to α_m. From the infinite variety of sets of α_m^* values satisfying this condition, a set is chosen such that the first sum on the right of eq (6) can be evaluated in closed form. If this technique is properly applied, the residual sum of the differences $[\alpha_m(j) - \alpha_m^*(j)] k^{4m}$ converges much more rapidly than the original sum in $\alpha_m(j) k^{4m}$.

For the present problem, it is shown in appendix A that by substituting Stirling's formula [11, p. 5] for the factorials in eq (5) such a set of $\alpha_m^*(j)$'s can be generated, of the form

$$\alpha_m^*(j) = A_j \left\{ \sum_{i \geq 0}^{j-1} B_{ij} \binom{m+j-i-1}{m} + \frac{1}{m+1} \right\} \tag{8}$$

where

$$A_j = (1/\sqrt{2}) 2^{-3j} \binom{2j}{j} \tag{9}$$

and

$$B_{ij} = (-1)^{j+i} 2^{2j-2i} \binom{j}{i} \Big/ \left[\binom{2j-2i}{j-i} \binom{j-\frac{1}{2}}{2j-2i} (j-i) \right]. \tag{10}$$

The notation $\sum_{i \geq 0}^{j-1}$ means that the finite sum over i vanishes when $j=0$.

Using (8), sums over $\alpha_m^*(j) k^{4m}$ can then be identified with the closed-form expressions

$$\sum_{m=0}^{\infty} \alpha_m^*(j) k^{4m} = A_j \left\{ \sum_{i \geq 0}^{j-1} B_{ij} [1/(1-k^4)]^{j-i} + (-1/k^4) \ln (1-k^4) \right\}. \tag{11}$$

From (8) we have, for $j=0$, 1 and 2

$$\alpha_m^*(0) = (1/\sqrt{2})\, \frac{1}{m+1}, \tag{12}$$

$$\alpha_m^*(1)=(1/\sqrt{2})\left\{4+(1/4)\ \frac{1}{m+1}\right\}, \tag{13}$$

and

$$\alpha_m^*(2)=(1/\sqrt{2})\left\{(16/3)(m+1)-1+(3/32)\ \frac{1}{m+1}\right\}. \tag{14}$$

The corresponding sums from (11) are

$$\sum_{m=0}^{\infty}\alpha_m^*(0)k^{4m}=(1/\sqrt{2})(-1/k^4)\ \ln\ (1-k^4), \tag{15}$$

$$\sum_{m=0}^{\infty}\alpha_m^*(1)k^{4m}=(1/\sqrt{2})\{4/(1-k^4)+(1/4)(-1/k^4)\ \ln\ (1-k^4)\}, \tag{16}$$

and

$$\sum_{m=0}^{\infty}\alpha_m^*(2)k^{4m}=(1/\sqrt{2})\{(16/3)/(1-k^4)^2-1/(1-k^4)+(3/32)(-1/k^4)\ \ln\ (1-k^4)\}. \tag{17}$$

Using (5) and the particular form of $\alpha_m^*(j)$ given in (8), the residual sum in (6), viz.,

$$\sum_R=\sum_{m=0}^{\infty}[\alpha_m(j)-\alpha_m^*(j)]k^{4m}$$

converges as fast as $\sum_{m=0}^{\infty}[1/(m+1)^2]k^{4m}$. This may be contrasted with the simple power series (4) which only converges as fast as $\sum_{m=0}^{\infty}m^{j-1}k^{4m}$.

An additional bonus is noted for the higher values of j. While Σ_R for any given j varies by less than a factor of two over the entire range $0\leq k^2\leq 1$, the total value of the function $\Omega_j(k)$ varies from π up to infinity over this same range. Thus, in the region of weakest convergence of Σ_R, i.e., $0.90\lesssim k^2\leq 1$, the contribution of Σ_R to $\Omega_j(k)$ may be neglected in many cases.

Also, in the right hand side of (11), the successive terms in the finite sum decrease very rapidly with increasing i, and the log-term, $(-1/k^4)\ \ln\ (1-k^4)$, is, for $j\geq 1$, $0\leq k^2\leq 1$, of the same order of magnitude or less than the final term $(i=j-1)$ in the finite sum. Thus, for example, for $j\geq 6$, $k^2\geq 0.90$, values of $\Omega_j(k)$ accurate to one part in 10^8 may be computed by omitting from (11) the finite series terms for $i\geq 5$, the log-term $(-1/k^4)\ \ln\ (1-k^4)$, and the entire sum Σ_R on the right side of (6).

This ease of attaining $\Omega_j(k)$ values of high precision for large j proved useful in evaluating disk-source Legendre coefficients [1] for large l, which involve smaller and smaller differences between larger and larger numbers.

2.3. $\Omega_j(k)$ in Terms of Bessel Functions, $j\geq 0$

The differencing technique illustrated for the sum in section 2.2 above may often profitably be applied to integrals as well, i.e., by writing

$$\int_a^b f(\phi)d\phi=\int_a^b f^*(\phi)d\phi+\int_a^b[f(\phi)-f^*(\phi)]d\phi. \tag{18}$$

In this expression $f^*(\phi)$ is a function, integrable in closed form over the region $a\leq\phi\leq b$, which is a suitable approximation to $f(\phi)$. Applied to $\Omega_j(k)$, it will be noted from (1−1) that

$$f(\phi)=(1-k^2\cos\ \phi)^{-n} \tag{19}$$

where

$$n=j+\tfrac{1}{2} \tag{20}$$

and

$$a=0,\ b=\pi. \tag{21}$$

For this case, it is convenient to take

$$f^*(\phi)=e^{+n(k^2\cos\phi)}+(n/2)(k^2\cos\phi)^2e^{+(n+2/3)(k^2\cos\phi)}, \tag{22}$$

an expression derived by retaining terms through $(k^2\cos\phi)^3$ in the expansion of $f(\phi)$ as given by (19). Then

$$\int_0^\pi f^*(\phi)d\phi=\int_0^\pi \exp\,[+n^2(k^2\cos\phi)]d\phi+(n/2)k^2\int_0^\pi(\cos^2\phi)\,\exp\left[\left(n+\frac{2}{3}\right)(k^2\cos\phi)\right]d\phi. \tag{23}$$

Now from the expression [12, p. 79, No. 9]

$$I_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi(\sin^2\phi)^\nu[\pm z\cos\phi]d\phi, \tag{24}$$

where $I_\nu(z)$ is the Bessel function of order ν with a purely imaginary argument, i.e.,

$$I_\nu(z)\equiv i^{-\nu}J_\nu(iz). \tag{25}$$

From (24)

$$I_0(z)=(1/\pi)\int_0^\pi\exp\,[\pm z\cos\phi]d\phi \tag{26a}$$

and

$$I_1(z)=(z/\pi)\int_0^\pi(\sin^2\phi)\,\exp\,[\pm z\cos\phi]d\phi. \tag{26b}$$

Using (26) to evaluate the integrals in (23), it is found that

$$\begin{aligned}(1/\pi)\Omega_j(k)=&+I_0[k^2(j+\tfrac{1}{2})]-(\tfrac{3}{2})k^2\left(\frac{2j+1}{6j+7}\right)I_1[k^2(j+\tfrac{7}{6})]+(\tfrac{1}{4})k^4(2j+1)I_0[k^2(j+\tfrac{7}{6})]\\&+(1/\pi)\int_0^\pi[(1-k^2\cos\phi)^{-(j+\frac{1}{2})}-\exp\,\{+(j+\tfrac{1}{2})(k^2\cos\phi)\}\\&-(\tfrac{1}{2})(j+\tfrac{1}{2})(k^2\cos\phi)^2\exp\{(j+\tfrac{7}{6})(k^2\cos\phi)\}+\ldots]d\phi.\end{aligned} \tag{27}$$

By expanding

$$(1-x)^{-n}-e^{+nx}-\tfrac{1}{2}nx^2e^{+(n+\frac{1}{2})x}$$

in a power series in $x=k^2\cos\phi$ and integrating term by term, the integral in (27) may be shown to be equal to

$$\begin{aligned}\sum_{m=2}^{\infty}(1/m!)^2\left[+\frac{(2j+4m)!}{(j+2m)!}\cdot\frac{j!}{(2j)!}-4^m(2j+1)^{2m}-18(2/3)^{2m}\cdot m(2m-1)(2j+1)(6j+7)^{2m-2}\right](k^4/64)^m\\=(3/64)\left(j+\frac{1}{2}\right)\left(j+\frac{29}{18}\right)k^8+\ldots.\end{aligned} \tag{28}$$

Further manipulation and simplification in the case of (28) is possible, but these procedures have not proven particularly rewarding. The greatest importance of eq (27), which it will be observed is exact, arises from the relation shown by this equation between $\Omega_j(k)$ and the Bessel functions. For numerical calculation, forms other than (27) and (28) may be more convenient, except possibly for some special cases.

It will be noted that $f^*(\phi)$ as given by (22) is a good approximation to $f(\phi)$ only for small values of k^2, since it is derived from the first three terms of the power series expansion of (19). Consequently, (27) is essentially useful only for $|k|<<1$. To realize the full potentialities of the formulation (18), it would be necessary to find a form for $f^*(\phi)$ valid over the whole range $0<k<1$ and integrable in closed form. Such an approximation is explored in section 2.5 below.

2.4. $\Omega_j(k)$ in Terms of Error Functions (the Method of Steepest Descent)

By writing

$$\Omega_j(k) \equiv \int_0^\pi e^{-F(\phi)} d\phi$$

so that

$$F(\phi) = (j+\tfrac{1}{2}) \ln (1-k^2 \cos \phi) \tag{29}$$

it is possible to obtain another interesting form for the function. Expanding $F(\phi)$ in a Maclaurin series [9, par. 6.104], it can readily be shown that all the odd derivatives of $F(\phi)$ vanish at the origin and that

$$\Omega_j(k) = e^{-F(0)} \int_0^\pi e^{-(\lambda/\pi)^2\phi^2} [1-(1/4!)F^{(4)}(0)\phi^4 - (1/6!)F^{(6)}(0)\phi^6 + \ldots] d\phi$$

where

$$(\lambda/\pi)^2 = (1/2)F^{(2)}(0). \tag{30}$$

Inserting the value of $F(0)$ and making the transformation $t = (\lambda/\pi)\phi$, it follows at once that

$$\Omega_j(k) = (\pi/\lambda)(1-k^2)^{-(j+\frac{1}{2})}[L_0(\lambda) - (1/4!)F^{(4)}(0)(\pi/\lambda)^4 L_2(\lambda) - (1/6!)F^{(6)}(0)(\pi/\lambda)^6 L_3(\lambda) + \ldots] \tag{31}$$

where

$$L_p(x) \equiv \int_0^x t^{2p} e^{-t^2} dt. \tag{32}$$

The integrals $L_p(x)$ can be expressed in terms of the incomplete gamma [13] or factorial [11] functions, $L_p(x) = \frac{1}{2} \cdot (p-\frac{1}{2}, x^2)!$ but an alternate procedure is perhaps more informative. Integrating (32) by parts, from the resulting recursion formula

$$L_{p+1}(x) = (1/2)[(2p+1)L_p(x) - x^{2p+1}e^{-x^2}] \tag{33}$$

the values of $L_p(x)$ can all be found in terms of $L_0(x)$. But this is simply

$$L_0(x) = (\sqrt{\pi}/2) \operatorname{erf} x \tag{34}$$

where the error function [5, par. 5.90] is

$$\operatorname{erf} x \equiv (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt.$$

By this method then, it is found that

$$L_2(x) = (3/4)\left[L_0(x) - xe^{-x^2}\left(1+\frac{2}{3}x^2\right)\right] \tag{35a}$$

$$L_3(x) = (15/8)\left[L_0(x) - xe^{-x^2}\left(1+\frac{2}{3}x^2+\frac{4}{15}x^4\right)\right]. \tag{35b}$$

The higher derivatives of $F(\phi)$ at the origin, $F^{(p)}(0)$, are readily determined by expanding $F(\phi)$ in a power series and recognizing that $F^{(p)}(0)$ is $p!$ times the coefficient of ϕ^p. In this way, it is found that

$$F^{(2)}(0) = +(j+\tfrac{1}{2})k^2(1-k^2)^{-1} \tag{36a}$$

$$F^{(4)}(0) = -(j+\tfrac{1}{2})k^2(1-k^2)^{-2}[2k^2+1] \tag{36b}$$

$$F^{(6)}(0) = +(j+\tfrac{1}{2})k^2(1-k^2)^{-3}[16k^4+13k^2+1]. \tag{36c}$$

From (36A) and (30),

$$\lambda = (\pi/2)\sqrt{(2j+1)k^2/(1-k^2)} \tag{37}$$

and combining this and the previous results with (31), it follows that

$$\Omega_j(k)=\sqrt{\pi/(2j+1)k^2}(1-k^2)^{-j}\Big[+\operatorname{erf}\lambda+(1/2)(2j+1)^{-1}\Big(1+\frac{1}{2k^2}\Big)\Big\{\operatorname{erf}\lambda-(2/\sqrt{\pi})(\lambda e^{-\lambda^2})\Big(1+\frac{2}{3}\lambda^2\Big)\Big\}$$
$$-(1/12)(2j+1)^{-2}\Big(16+\frac{13}{k^2}+\frac{1}{k^4}\Big)\Big\{\operatorname{erf}\lambda-(2/\sqrt{\pi})(\lambda e^{-\lambda^2})\Big(1+\frac{2}{3}\lambda^2+\frac{4}{15}\lambda^4\Big)\Big\}+\ldots\Big]. \quad (38)$$

In using these relations for small k^2 or λ values, the problem of small differences between large quantities is resolved by the series

$$\{\operatorname{erf}\lambda-(2/\sqrt{\pi})(\lambda e^{-\lambda^2})(1+\tfrac{2}{3}\lambda^2)\}=(2/\sqrt{\pi})(4\lambda^5/15)\{1-(5/7)\lambda^2+(5/18)\lambda^4-(5/66)\lambda^6$$
$$+(5/312)\lambda^8-(1/360)\lambda^{10}+(1/2448)\lambda^{12}+\ldots\} \quad (39)$$

and

$$\Big\{\operatorname{erf}\lambda-(2/\sqrt{\pi})(\lambda e^{-\lambda^2})\Big(1+\frac{2}{3}\lambda^2+\frac{4}{15}\lambda^4\Big)\Big\}$$
$$=(2/\sqrt{\pi})(8\lambda^7/105)\{1+(77/16)\lambda^2+(7/22)\lambda^4-(7/78)\lambda^6+(7/360)\lambda^8-(7/2040)\lambda^{10}+\ldots\}. \quad (40)$$

For large λ values, i.e., large j's, erf λ and the bracket terms involving this quantity are all close to unity, and

$$\lim_{j\to\infty}\Omega_j(k)=\sqrt{\pi/(2j+1)k^2}(1-k^2)^{-j}\Big[1+(1/2)(2j+1)^{-1}\Big\{1+\frac{1}{2k^2}\Big\}$$
$$-(4/3)(2j+1)^{-2}\Big\{1+\frac{13}{16k^2}+\frac{1}{16k^4}\Big\}+\ldots\Big]. \quad (41)$$

2.5. Numerical Integration Techniques for $\Omega_j(k)$

To apply eq (18) in a more general way than was used in section 2.3 above, let

$$f^*(\phi)=(1-k^2)^{-n} \quad (42)$$

where $n=j+\frac{1}{2}$, as before (eq 20). Then from (18)

$$\Omega_j(k)=\pi(1-k^2)^{-n}+\int_0^\pi[(1-k^2\cos\phi)^{-n}-(1-k^2)^{-n}]d\phi.$$

It will be observed that the integrand here contains no singularities, even at the origin for $k=1$. With the change of variable $\phi=\pi x$, this becomes

$$\Omega_j(k)=\pi(1-k^2)^{-n}+\pi\int_0^1[\{1-k^2\cos(\pi x)^{-n}\}-(1-k^2)^{-n}]dx.$$

This integral is now in standard form for the use of numerical integration techniques. By fitting a polynomial of the N'th degree to $\Xi(x)$, it can be shown that

$$\int_0^1\Xi(x)dx\cong\sum_{i=0}^N w_N(i)\Xi(i/N)\Big/\sum_{i=0}^N w_N(i) \quad (43)$$

where the $w_N(i)$'s are constants (given for example in the introduction to the *Tables of Lagrangian Interpolation Coefficients* [14] where the integrals are expressed in slightly different form). It should be noted that (43) can never be expected to give exact answers; but by increasing the value of N, closer and closer approximations to $\Omega_j(k)$ can be obtained over the whole range of k, although this procedure does not necessarily result in improved accuracy at any given value of k. In the present case, where for certain values of N, cos $(i\pi/N)$ can be expressed in simple closed form, this technique leads to an interesting series of approximations

for $\Omega_j(k)$ as given below. Note that all these results yield exactly the right answers at $k=0$, $\Omega_j(0)=\pi$

$$n=j+\frac{1}{2}$$

$$N=2$$

$$\Omega_j(k)=(\pi/6)[+4+(1-k^2)^{-n}+(1+k^2)^{-n}] \tag{44}$$

$$N=3$$

$$\Omega j(k)=(\pi/8)\left[+(1-k^2)^{-n}+(1+k^2)^{-n}+3\left(1-\frac{1}{2}k^2\right)^{-n}+3\left(1+\frac{1}{2}k^2\right)^{-n}\right] \tag{45}$$

$$N=4$$

$$\Omega_j(k)=(\pi/90)\left[+12+7(1-k^2)^{-n}+7(1+k^2)^{-n}+32\left(1-\frac{1}{\sqrt{2}}k^2\right)^{-n}+32\left(1+\frac{1}{\sqrt{2}}k^2\right)^{-n}\right] \tag{46}$$

$$N=5$$

$$\Omega_j(k)=(\pi/288)\left[+19(1-k^2)^{-n}+19(1+k^2)^{-n}+75\left(1-\frac{\sqrt{5}+1}{4}k^2\right)^{-n}\right.$$
$$\left.+75\left(1+\frac{\sqrt{5}+1}{4}k^2\right)^{-n}+50\left(1-\frac{\sqrt{5}-1}{4}k^2\right)^{-n}+50\left(1+\frac{\sqrt{5}-1}{4}k^2\right)^{-n}\right] \tag{47}$$

$$N=6$$

$$\Omega_j(k)=(\pi/840)[+272+41(1-k^2)^{-n}+41(1+k^2)^{-n}+216(1-\tfrac{1}{2}\sqrt{3}k^2)^{-n}$$
$$+216(1+\tfrac{1}{2}\sqrt{3}k^2)^{-n}+27(1-\tfrac{1}{2}k^2)^{-n}+27(1+\tfrac{1}{2}k^2)^{-n}] \tag{48}$$

$$N=10$$

$$\Omega_j(k)=(\pi/598{,}752)\left[+427{,}368+16{,}067(1-k^2)^{-n}+16{,}067(1+k^2)^{-n}\right.$$
$$+106{,}300\left(1-\sqrt{\frac{5+\sqrt{5}}{8}}k^2\right)^{-n}+106{,}300\left(1+\sqrt{\frac{5+\sqrt{5}}{8}}k^2\right)^{-n}-48{,}525\left(1-\frac{\sqrt{5}+1}{4}k^2\right)^{-n}$$
$$-48{,}525\left(1+\frac{\sqrt{5}+1}{4}k^2\right)^{-n}+272{,}400\left(1-\sqrt{\frac{5-\sqrt{5}}{8}}k^2\right)^{-n}+272{,}400\left(1+\sqrt{\frac{5-\sqrt{5}}{8}}k^2\right)^{-n}$$
$$\left.-260{,}550\left(1-\frac{\sqrt{5}-1}{4}k^2\right)^{-n}-260{,}550\left(1+\frac{\sqrt{5}-1}{4}k^2\right)^{-n}\right]. \tag{49}$$

The interesting and beautiful symmetries in these approximations have been emphasized in the formulas above. It is evident that by using a more complicated expression for $f^*(\phi)$ than (42), better agreement can be achieved. Also, by going to larger values of N, or by dividing the range of ϕ into smaller intervals, formulas of greater accuracy can be developed. For large values of j and k^2, or their product

$$\lim_{jk^2\to\infty}\Omega_j(k)=\pi(1-k^2)^{-(j+\frac{1}{2})}w_N(0)\Big/\sum_{i=0}^{N}w_N(i)$$

or inserting the expression from $w_N(i)$ given by Whittaker and Robinson [15]

$$\lim_{jk^2\to\infty} \Omega_j(k)=\pi(1-k^2)^{-(j+\frac{1}{2})}\cdot\frac{(-1)^N}{N}\int_0^N\binom{t-1}{N}dt \tag{50}$$

where, it will turn out, since the $w_N(i)$'s are all integers, the last term here is a rational fraction.

3. Numerical

It is immediately apparent from the definition of the function that

$$\Omega_j(0)=\pi \text{ and } \Omega_j(1)=\infty. \tag{1}$$

The intermediate behavior of $\Omega_j(k)$ versus k^2 is displayed for $0\leq j(1)\leq 9$ as a family of curves in figure 1. These curves are based on values of $\Omega_j(k)$ listed for $0\leq j(1)\leq 9$ and $0\leq k^2(0.01)\leq 0.99$ in table 1.

Table 1 was computed in "double precision" on the National Bureau of Standards IBM 7090 electronic computing machine, using the difference-sum (2.2–6) with the particular $\alpha_m^*(j)$ given in (2.2–8). The results are given to eight digits, which were rounded off from a table correct to at least nine digits throughout, so all eight digits can be considered significant. The results in the region of poorest convergence (high k, low j) of the difference sum were verified by a ten-digit hand calculation using the recursion expression (1–7) and the high-k elliptic integral expansions (1–10) and (1–11).

The lengthy developments in section 2 can be considered to yield various approximations to the value of $\Omega_j(k)$. In tables 2 to 6 numerical examples are presented which will give some idea of the usefulness of the techniques. For each example, the values $j=0$ and 8, and $k^2=0.01$ and 0.99 are used for purposes of illustration.

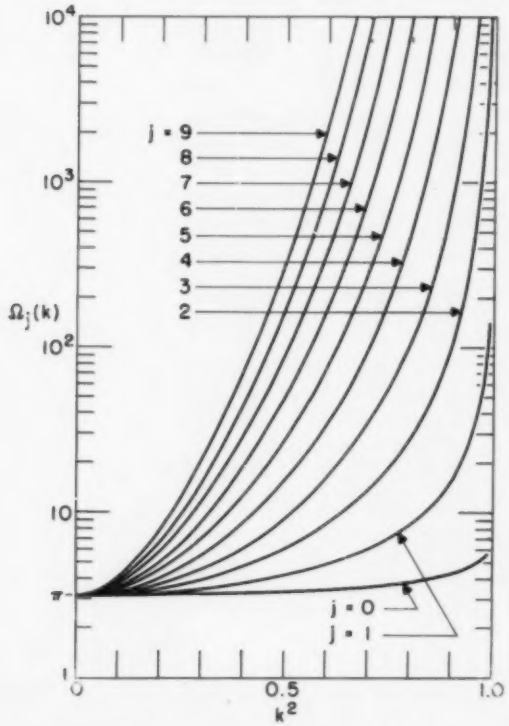

FIGURE 1. *Family of* $\Omega_j(k)$ *curves based on numerical results in table 1.*

TABLE 1.* *Values of* $\Omega_j(k)$ *for* $0 \leq j(1) \leq 9$ *and* $0 \leq k^2\ (0.01) < 1.00$

These were computed using the difference sum (2.2−6) with the $a_m^*(j)$ in (2.2−8). A tolerance of $<10^{-11}$ was set on the fractional contribution of the first neglected series term $|a_m^*(j) - a_m^*(j)|k^{2m}$ which resulted in nine or more significant figures throughout, here rounded off to eight significant figures.

k^2	Ω_0	Ω_1	Ω_2	Ω_3	Ω_4
0.00	π	π	π	π	π
.01	3.1416516 (0)	3.1418872 (0)	3.1422800 (0)	3.1428299 (0)	3.1435371 (0)
.02	3.1418283 (0)	3.1427712 (0)	3.1443432 (0)	3.1465451 (0)	3.1493776 (0)
.03	3.1421231 (0)	3.1442457 (0)	3.1477863 (0)	3.1527481 (0)	3.1591355 (0)
.04	3.1425360 (0)	3.1463125 (0)	3.1526155 (0)	3.1614556 (0)	3.1728466 (0)
.05	3.1430673 (0)	3.1489739 (0)	3.1588399 (0)	3.1726912 (0)	3.1905614 (0)
.06	3.1437174 (0)	3.1522332 (0)	3.1664711 (0)	3.1864852 (0)	3.2123454 (0)
.07	3.1444868 (0)	3.1560943 (0)	3.1755235 (0)	3.2028752 (0)	3.2382796 (0)
.08	3.1453758 (0)	3.1605617 (0)	3.1860142 (0)	3.2219059 (0)	3.2684609 (0)
.09	3.1463852 (0)	3.1656409 (0)	3.1979629 (0)	3.2439296 (0)	3.3030029 (0)
.10	3.1475156 (0)	3.1713379 (0)	3.2113926 (0)	3.2691065 (0)	3.3420368 (0)
.11	3.1487677 (0)	3.1776597 (0)	3.2263290 (0)	3.2954049 (0)	3.3857120 (0)
.12	3.1501424 (0)	3.1846141 (0)	3.2428009 (0)	3.3256017 (0)	3.4341976 (0)
.13	3.1516407 (0)	3.1922094 (0)	3.2608406 (0)	3.3587827 (0)	3.4876832 (0)
.14	3.1532635 (0)	3.2004553 (0)	3.2804834 (0)	3.3950431 (0)	3.5463805 (0)
.15	3.1550119 (0)	3.2093619 (0)	3.3017684 (0)	3.4344882 (0)	3.6105249 (0)
.16	3.1568872 (0)	3.2189405 (0)	3.3247382 (0)	3.4772340 (0)	3.6803773 (0)
.17	3.1588907 (0)	3.2292032 (0)	3.3494395 (0)	3.5234078 (0)	3.7562257 (0)
.18	3.1610237 (0)	3.2401632 (0)	3.3759229 (0)	3.5731488 (0)	3.8383882 (0)
.19	3.1632878 (0)	3.2518347 (0)	3.4042436 (0)	3.6266094 (0)	3.9272147 (0)
.20	3.1656846 (0)	3.2642330 (0)	3.4344610 (0)	3.6839559 (0)	4.0230902 (0)
.21	3.1682158 (0)	3.2773746 (0)	3.4666396 (0)	3.7453695 (0)	4.1264381 (0)
.22	3.1708833 (0)	3.2912770 (0)	3.5008493 (0)	3.8110474 (0)	4.2377233 (0)
.23	3.1736890 (0)	3.3059501 (0)	3.5371653 (0)	3.8812046 (0)	4.3574565 (0)
.24	3.1766349 (0)	3.3214413 (0)	3.5756986 (0)	3.9560749 (0)	4.4861987 (0)
.25	3.1797233 (0)	3.3377450 (0)	3.6164468 (0)	4.0359125 (0)	4.6245662 (0)
.26	3.1829566 (0)	3.3548935 (0)	3.6595943 (0)	4.1209941 (0)	4.7732360 (0)
.27	3.1863371 (0)	3.3729113 (0)	3.7052127 (0)	4.2116207 (0)	4.9329530 (0)
.28	3.1898676 (0)	3.3918249 (0)	3.7534115 (0)	4.3081200 (0)	5.1045360 (0)
.29	3.1935508 (0)	3.4116624 (0)	3.8043086 (0)	4.4108487 (0)	5.2888871 (0)
.30	3.1973897 (0)	3.4324538 (0)	3.8580313 (0)	4.5201954 (0)	5.4869698 (0)
.31	3.2013874 (0)	3.4542312 (0)	3.9147167 (0)	4.6365838 (0)	5.6999702 (0)
.32	3.2055471 (0)	3.4770287 (0)	3.9745127 (0)	4.7604758 (0)	5.9290085 (0)
.33	3.2098724 (0)	3.5008829 (0)	4.0375787 (0)	4.8923759 (0)	6.1754521 (0)
.34	3.2143669 (0)	3.5258327 (0)	4.1040871 (0)	5.0328355 (0)	6.4407815 (0)
.35	3.2190345 (0)	3.5519197 (0)	4.1742238 (0)	5.1824573 (0)	6.7266372 (0)
.36	3.2238793 (0)	3.5791883 (0)	4.2481901 (0)	5.3419016 (0)	7.0348393 (0)
.37	3.2289056 (0)	3.6076862 (0)	4.3262036 (0)	5.5118921 (0)	7.3674111 (0)
.38	3.2341182 (0)	3.6374640 (0)	4.4085000 (0)	5.6932230 (0)	7.7266040 (0)
.39	3.2395217 (0)	3.6685762 (0)	4.4953348 (0)	5.8867672 (0)	8.1149281 (0)
.40	3.2451213 (0)	3.7010809 (0)	4.5868850 (0)	6.0934851 (0)	8.5351864 (0)
.41	3.2509225 (0)	3.7350405 (0)	4.6837518 (0)	6.3144352 (0)	8.9905145 (0)
.42	3.2569310 (0)	3.7705219 (0)	4.7859626 (0)	6.5507857 (0)	9.4844263 (0)
.43	3.2631530 (0)	3.8075968 (0)	4.8939738 (0)	6.8038277 (0)	1.0020868 (1)
.44	3.2695947 (0)	3.8463422 (0)	5.0081739 (0)	7.0749911 (0)	1.0604277 (1)
.45	3.2762632 (0)	3.8868412 (0)	5.1289871 (0)	7.3658616 (0)	1.1239660 (1)
.46	3.2831656 (0)	3.9291828 (0)	5.2568770 (0)	7.6782009 (0)	1.1932671 (1)
.47	3.2903097 (0)	3.9734632 (0)	5.3923515 (0)	8.0139705 (0)	1.2689713 (1)
.48	3.2977036 (0)	4.0197860 (0)	5.5359670 (0)	8.3753580 (0)	1.3518054 (1)
.49	3.3053561 (0)	4.0682633 (0)	5.6883349 (0)	8.7648084 (0)	1.4425963 (1)
.50	3.3132763 (0)	4.1190161 (0)	5.8501280 (0)	9.1850601 (0)	1.5422873 (1)
.51	3.3214743 (0)	4.1721755 (0)	6.0220875 (0)	9.6391874 (0)	1.6519570 (1)
.52	3.3299604 (0)	4.2278838 (0)	6.2050823 (0)	1.0130649 (1)	1.7728428 (1)
.53	3.3387459 (0)	4.2862954 (0)	6.3998682 (0)	1.0663346 (1)	1.9063672 (1)
.54	3.3478428 (0)	4.3475784 (0)	6.6075997 (0)	1.1241689 (1)	2.0541717 (1)
.55	3.3572639 (0)	4.4119159 (0)	6.8293426 (0)	1.1870679 (1)	2.2181554 (1)
.56	3.3670230 (0)	4.4795076 (0)	7.0663401 (0)	1.2556002 (1)	2.4005228 (1)
.57	3.3771348 (0)	4.5505724 (0)	7.3199796 (0)	1.3304138 (1)	2.6038421 (1)
.58	3.3876151 (0)	4.6253495 (0)	7.5918137 (0)	1.4122502 (1)	2.8311150 (1)
.59	3.3984809 (0)	4.7041020 (0)	7.8835849 (0)	1.5019596 (1)	3.0858639 (1)
.60	3.4097506 (0)	4.7871193 (0)	8.1972534 (0)	1.6005209 (1)	3.3722376 (1)
.61	3.4214439 (0)	4.8747204 (0)	8.5350309 (0)	1.7090647 (1)	3.6951427 (1)
.62	3.4335822 (0)	4.9672582 (0)	8.8994209 (0)	1.8286016 (1)	4.0604078 (1)
.63	3.4461887 (0)	5.0651235 (0)	9.2932656 (0)	1.9615571 (1)	4.4749869 (1)
.64	3.4592883 (0)	5.1687509 (0)	9.7198032 (0)	2.1088134 (1)	4.9472173 (1)
.65	3.4729085 (0)	5.2786244 (0)	1.0182736 (1)	2.2727623 (1)	5.4871440 (1)
.66	3.4870789 (0)	5.3952852 (0)	1.0686311 (1)	2.4558693 (1)	6.1069344 (1)
.67	3.5018320 (0)	5.5193397 (0)	1.1235424 (1)	2.6610551 (1)	6.8214090 (1)
.68	3.5172034 (0)	5.6514705 (0)	1.1835738 (1)	2.8917967 (1)	7.6487274 (1)
.69	3.5332319 (0)	5.7924479 (0)	1.2493834 (1)	3.1522555 (1)	8.6112803 (1)
.70	3.5499606 (0)	5.9431449 (0)	1.3217398 (1)	3.4474411 (1)	9.7368612 (1)

See footnote at end of table, p. 12.

TABLE 1.* *Values of* $\Omega_j(k)$ *for* $0 \leq j(1) \leq 9$ *and* $0 \leq k^2\ (0.01) < 1.00$—Continued

k^2	Ω_0	Ω_1	Ω_2	Ω_3	Ω_4
.71	3. 5674369 (0)	6. 1045547 (0)	1. 4015448 (1)	3. 7834208 (1)	1. 1060216 (2)
.72	3. 5857131 (0)	6. 2778123 (0)	1. 4898627 (1)	4. 1675905 (1)	1. 2625110 (2)
.73	3. 6048476 (0)	6. 4642206 (0)	1. 5879565 (1)	4. 6090284 (1)	1. 4487127 (2)
.74	3. 6249055 (0)	6. 6652823 (0)	1. 6973345 (1)	5. 1189617 (1)	1. 6717465 (2)
.75	3. 6459595 (0)	6. 8827406 (0)	1. 8198097 (1)	5. 7113855 (1)	1. 9468189 (2)
.76	3. 6680915 (0)	7. 1186284 (0)	1. 9575775 (1)	6. 4039975 (1)	2. 2679546 (2)
.77	3. 6913941 (0)	7. 3753315 (0)	2. 1133163 (1)	7. 2188313 (1)	2. 6990316 (2)
.78	3. 7159722 (0)	7. 6556690 (0)	2. 2903221 (1)	8. 1848192 (1)	3. 1652641 (2)
.79	3. 7419457 (0)	7. 9629956 (0)	2. 4926875 (1)	9. 3389738 (1)	3. 7853627 (2)
.80	3. 7694524 (0)	8. 3013360 (0)	2. 7255455 (1)	1. 0729980 (2)	4. 5687313 (2)
.81	3. 7986513 (0)	8. 6755906 (0)	2. 9954048 (1)	1. 2422547 (2)	5. 5702880 (2)
.82	3. 8297278 (0)	9. 0916202 (0)	3. 3106179 (1)	1. 4503942 (2)	6. 8678797 (2)
.83	3. 8628993 (0)	9. 5568633 (0)	3. 6820480 (1)	1. 7093748 (2)	8. 5739421 (2)
.84	3. 8984232 (0)	1. 0080473 (1)	4. 1240339 (1)	2. 0358783 (2)	1. 0854291 (3)
.85	3. 9366074 (0)	1. 0674075 (1)	4. 6558190 (1)	2. 4536454 (2)	1. 3959246 (3)
.86	3. 9778240 (0)	1. 1352615 (1)	5. 3037170 (1)	2. 9972313 (2)	1. 8276781 (3)
.87	4. 0225290 (0)	1. 2135628 (1)	6. 1044816 (1)	3. 7182365 (2)	2. 4426513 (3)
.88	4. 0712886 (0)	1. 3049152 (1)	7. 1107150 (1)	4. 6960084 (2)	3. 3432588 (3)
.89	4. 1248185 (0)	1. 4128679 (1)	8. 3998552 (1)	6. 0567809 (2)	4. 7095582 (3)
.90	4. 1840408 (0)	1. 5423880 (1)	1. 0089734 (2)	8. 0085478 (2)	6. 8473464 (3)
.91	4. 2501717 (0)	1. 7006477 (1)	1. 2366829 (2)	1. 0917125 (3)	1. 0373314 (4)
.92	4. 3248613 (0)	1. 8984009 (1)	1. 5540620 (2)	1. 5446583 (3)	1. 6516804 (4)
.93	4. 4104268 (0)	2. 1525401 (1)	2. 0155731 (2)	2. 2914615 (3)	2. 8010734 (4)
.94	4. 5102677 (0)	2. 4912083 (1)	2. 7244578 (2)	3. 6165464 (3)	5. 1390976 (4)
.95	4. 6296635 (0)	2. 9650431 (1)	3. 8964807 (2)	6. 2117605 (3)	1. 0636320 (5)
.96	4. 7774745 (0)	3. 6752779 (1)	6. 0473488 (2)	1. 2060257 (4)	2. 5819863 (5)
.97	4. 9703403 (0)	4. 8580263 (1)	1. 0679679 (3)	2. 8419636 (4)	8. 1144742 (5)
.98	5. 2456872 (0)	7. 2213223 (1)	2. 3872660 (3)	9. 5361054 (4)	4. 0851238 (6)
.99	5. 7231768 (0)	1. 4303674 (2)	9. 4878355 (3)	7. 5852838 (5)	6. 5002882 (7)
1. 00	∞	∞	∞	∞	∞

k^2	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
0. 00	π	π	π	π	π
.01	3. 1444016 (0)	3. 1454234 (0)	3. 1466027 (0)	3. 1479395 (0)	3. 1494340 (0)
.02	3. 1528418 (0)	3. 1569389 (0)	3. 1616703 (0)	3. 1670377 (0)	3. 1730429 (0)
.03	3. 1669537 (0)	3. 1762090 (0)	3. 1869086 (0)	3. 1990608 (0)	3. 2126747 (0)
.04	3. 1868051 (0)	3. 2033511 (0)	3. 2225075 (0)	3. 2443006 (0)	3. 2687599 (0)
.05	3. 2124917 (0)	3. 2385310 (0)	3. 2687360 (0)	3. 3031717 (0)	3. 3419109 (0)
.06	3. 2441379 (0)	3. 2819649 (0)	3. 3259458 (0)	3. 3762170 (0)	3. 4329324 (0)
.07	3. 2818977 (0)	3. 3339215 (0)	3. 3945750 (0)	3. 4641155 (0)	3. 5428347 (0)
.08	3. 3259565 (0)	3. 3947246 (0)	3. 4751540 (0)	3. 5676927 (0)	3. 6728507 (0)
.09	3. 3765321 (0)	3. 4647569 (0)	3. 5683119 (0)	3. 6879319 (0)	3. 8244570 (0)
.10	3. 4338769 (0)	3. 5444634 (0)	3. 6747845 (0)	3. 8259893 (0)	3. 9993993 (0)
.11	3. 4982801 (0)	3. 6343569 (0)	3. 7954235 (0)	3. 9832113 (0)	4. 1997232 (0)
.12	3. 5700701 (0)	3. 7350228 (0)	3. 9312075 (0)	4. 1611549 (0)	4. 4278102 (0)
.13	3. 6496177 (0)	3. 8471259 (0)	4. 0832553 (0)	4. 3616121 (0)	4. 6864215 (0)
.14	3. 7373392 (0)	3. 9714180 (0)	4. 2528403 (0)	4. 5869377 (0)	4. 9787483 (0)
.15	3. 8337006 (0)	4. 1087457 (0)	4. 4414078 (0)	4. 8385832 (0)	5. 3084726 (0)
.16	3. 9392217 (0)	4. 2600610 (0)	4. 6505957 (0)	5. 1201345 (0)	5. 6798373 (0)
.17	4. 0544809 (0)	4. 4264317 (0)	4. 8822565 (0)	5. 4343576 (0)	6. 0977301 (0)
.18	4. 1801214 (0)	4. 6090547 (0)	5. 1384852 (0)	5. 7847509 (0)	6. 5677814 (0)
.19	4. 3168573 (0)	4. 8092705 (0)	5. 4216493 (0)	6. 1753062 (0)	7. 0964805 (0)
.20	4. 4654806 (0)	5. 0285795 (0)	5. 7344248 (0)	6. 6105807 (0)	7. 6913118 (0)
.21	4. 6268698 (0)	5. 2686619 (0)	6. 0798376 (0)	7. 0957805 (0)	8. 3609171 (0)
.22	4. 8019991 (0)	5. 5313990 (0)	6. 4613109 (0)	7. 6368586 (0)	9. 1152864 (0)
.23	4. 9919487 (0)	5. 8188984 (0)	6. 8827211 (0)	8. 2406300 (0)	9. 9659856 (0)
.24	5. 1979173 (0)	6. 1335234 (0)	7. 3484615 (0)	8. 9149065 (0)	1. 0926426 (1)
.25	5. 4212351 (0)	6. 4779255 (0)	7. 8635180 (0)	9. 6686559 (0)	1. 2012186 (1)
.26	5. 6633798 (0)	6. 8550829 (0)	8. 4335554 (0)	1. 0512189 (1)	1. 3241396 (1)
.27	5. 9259941 (0)	7. 2683447 (0)	9. 0650197 (0)	1. 1457381 (1)	1. 4635194 (1)
.28	6. 2109057 (0)	7. 7214814 (0)	9. 7652570 (0)	1. 2517935 (1)	1. 6218282 (1)
.29	6. 5201504 (0)	8. 2187441 (0)	1. 0542653 (1)	1. 3709692 (1)	1. 8019586 (1)
.30	6. 8559986 (0)	8. 7649324 (0)	1. 1406796 (1)	1. 5051001 (1)	2. 0073050 (1)
.31	7. 2209852 (0)	9. 3654742 (0)	1. 2368671 (1)	1. 6563161 (1)	2. 2418611 (1)
.32	7. 6179448 (0)	1. 0026517 (1)	1. 3440887 (1)	1. 8270952 (1)	2. 5103359 (1)
.33	8. 0500508 (0)	1. 0755037 (1)	1. 4637943 (1)	2. 0203264 (1)	2. 8182967 (1)
.34	8. 5208624 (0)	1. 1558963 (1)	1. 5976553 (1)	2. 2393861 (1)	3. 1723419 (1)
.35	9. 0343771 (0)	1. 2447325 (1)	1. 7476017 (1)	2. 4882297 (1)	3. 5803135 (1)
.36	9. 5950927 (0)	1. 3430429 (1)	1. 9158678 (1)	2. 7715029 (1)	4. 0515567 (1)
.37	1. 0208078 (1)	1. 4520056 (1)	2. 1050462 (1)	3. 0946757 (1)	4. 5972404 (1)
.38	1. 0879058 (1)	1. 5729710 (1)	2. 3181518 (1)	3. 4642066 (1)	5. 2307520 (1)
.39	1. 1614509 (1)	1. 7074894 (1)	2. 5587002 (1)	3. 8877418 (1)	5. 9681868 (1)
.40	1. 2421771 (1)	1. 8573456 (1)	2. 8308011 (1)	4. 3743600 (1)	6. 8289570 (1)

See footnote at end of table, p. 12.

TABLE 1.* *Values of* $\Omega_j(k)$ *for* $0 \le j(1) \le 9$ *and* $0 \le k^2\ (0.01) < 1.00$—Continued

k^2	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
.41	1.3309183 (1)	2.0245991 (1)	3.1392711 (1)	4.9348723 (1)	7.8365504 (1)
.42	1.4286240 (1)	2.2116313 (1)	3.4897718 (1)	5.5821923 (1)	9.0194823 (1)
.43	1.5363771 (1)	2.4212044 (1)	3.8889761 (1)	6.3317936 (1)	1.0412490 (2)
.44	1.6554164 (1)	2.6565293 (1)	4.3447730 (1)	7.2022786 (1)	1.2058044 (2)
.45	1.7871617 (1)	2.9213497 (1)	4.8665169 (1)	8.2160859 (1)	1.4008261 (2)
.46	1.9332450 (1)	3.2200424 (1)	5.4653362 (1)	9.4003772 (1)	1.6327337 (2)
.47	2.0955464 (1)	3.5577398 (1)	6.1545121 (1)	1.0788151 (2)	1.9084667 (2)
.48	2.2762382 (1)	3.9404785 (1)	6.9499504 (1)	1.2419646 (2)	2.2408845 (2)
.49	2.4778370 (1)	4.3753812 (1)	7.8707665 (1)	1.4344125 (2)	2.6392830 (2)
.50	2.7032673 (1)	4.8708801 (1)	8.9400186 (1)	1.6622140 (2)	3.1200644 (2)
.51	2.9559372 (1)	5.4369916 (1)	1.0185627 (2)	1.9328437 (2)	3.7026116 (2)
.52	3.2398316 (1)	6.0856567 (1)	1.1641533 (2)	2.2556083 (2)	4.4114329 (2)
.53	3.5596245 (1)	6.8311633 (1)	1.3349170 (2)	2.6419288 (2)	5.2776727 (2)
.54	3.9208178 (1)	7.6906742 (1)	1.5359335 (2)	3.1063675 (2)	6.3411131 (2)
.55	4.3299101 (1)	8.6848881 (1)	1.7734580 (2)	3.6670460 (2)	7.6528445 (2)
.56	4.7946069 (1)	9.8388739 (1)	2.0552302 (2)	4.3469219 (2)	9.2788504 (2)
.57	5.3240797 (1)	1.1183126 (2)	2.3908727 (2)	5.1751711 (2)	1.1304851 (3)
.58	5.9292896 (1)	1.2754912 (2)	2.7924109 (2)	6.1890817 (2)	1.3842894 (3)
.59	6.6233924 (1)	1.4559691 (2)	3.2749532 (2)	7.4365906 (2)	1.7040380 (3)
.60	7.4222490 (1)	1.6774836 (2)	3.8575879 (2)	8.9797058 (2)	2.1092534 (3)
.61	8.3450710 (1)	1.9349511 (2)	4.5645739 (2)	1.0899157 (3)	2.6259746 (3)
.62	9.4152431 (1)	2.2411423 (2)	5.4269316 (2)	1.3300762 (3)	3.2891909 (3)
.63	1.0661378 (2)	2.6070254 (2)	6.4845852 (2)	1.6324220 (3)	4.1462859 (3)
.64	1.2118673 (2)	3.0464479 (2)	7.7892704 (2)	2.0155347 (3)	5.2619563 (3)
.65	1.3830982 (2)	3.5770065 (2)	9.4085123 (2)	2.5043262 (3)	6.7253039 (3)
.66	1.5851615 (2)	4.2212147 (2)	1.1431115 (3)	3.1324761 (3)	8.6601663 (3)
.67	1.8249392 (2)	5.0080856 (2)	1.3974809 (3)	3.9459207 (3)	1.1240327 (4)
.68	2.1109677 (2)	5.9752941 (2)	1.7197003 (3)	5.0079025 (3)	1.4712160 (4)
.69	2.4541303 (2)	7.1721622 (2)	2.1310065 (3)	6.4063588 (3)	1.9428762 (4)
.70	2.8683593 (2)	8.6638166 (2)	2.6603265 (3)	8.2648601 (3)	2.5802033 (4)
.71	3.3716364 (2)	1.0537046 (3)	3.3474663 (3)	1.0759010 (4)	3.4883257 (4)
.72	3.9873716 (2)	1.2808644 (3)	4.2478017 (3)	1.4141363 (4)	4.7489570 (4)
.73	4.7463268 (2)	1.5937432 (3)	5.4392698 (3)	1.8779832 (4)	6.5405516 (4)
.74	5.6893286 (2)	1.9841824 (3)	7.0320983 (3)	2.5217787 (4)	9.1209641 (4)
.75	6.8711446 (2)	2.4925823 (3)	9.1892247 (3)	3.4209872 (4)	1.2891309 (5)
.76	8.3660947 (2)	3.1618092 (3)	1.2143196 (4)	4.7175815 (4)	1.8486494 (5)
.77	1.0276297 (3)	4.0531621 (3)	1.6244726 (4)	6.5857893 (4)	2.6030537 (5)
.78	1.2743971 (3)	5.2556423 (3)	2.2023463 (4)	9.3349272 (4)	3.9909042 (5)
.79	1.5970122 (3)	6.9006308 (3)	3.0296111 (4)	1.3453630 (5)	6.0258827 (5)
.80	2.0243431 (3)	9.1856068 (3)	4.2347605 (4)	1.9746667 (5)	9.2871234 (5)
.81	2.5985857 (3)	1.2413348 (4)	6.0244685 (4)	2.9572117 (5)	1.4640725 (6)
.82	3.3826250 (3)	1.7058333 (4)	8.7393424 (4)	4.5284036 (5)	2.3665859 (6)
.83	4.4722115 (3)	2.3882318 (4)	1.2956045 (5)	7.1085875 (5)	3.9336838 (6)
.84	6.0166633 (3)	3.4141676 (4)	1.9680626 (5)	1.1473581 (6)	6.7461970 (6)
.85	8.2551518 (3)	4.9972061 (4)	3.0728282 (5)	1.9109387 (6)	1.1985330 (7)
.86	1.1582518 (4)	7.5129608 (4)	4.9500854 (5)	3.2984010 (6)	2.2165832 (7)
.87	1.6673365 (4)	1.1648160 (5)	8.2655169 (5)	5.9314923 (6)	4.2928225 (7)
.88	2.4726623 (4)	1.8715471 (5)	1.4388026 (6)	1.1186011 (7)	8.7708067 (7)
.89	3.7972735 (4)	3.1357024 (5)	2.6299573 (6)	2.2306362 (7)	1.9080277 (8)
.90	6.0789976 (4)	5.3223607 (5)	5.0951320 (6)	4.7538466 (7)	4.4730772 (8)
.91	1.0234053 (5)	1.0330799 (6)	1.0591208 (7)	1.0980174 (8)	1.1479956 (9)
.92	1.8334510 (5)	2.0822980 (6)	2.4017606 (7)	2.8013155 (8)	3.2950159 (9)
.93	3.5540054 (5)	4.6133600 (6)	6.0816052 (7)	8.1089617 (8)	1.0898261 (10)
.94	7.6378369 (5)	1.1567756 (7)	1.7791735 (8)	2.7670704 (9)	4.3398773 (10)
.95	1.8898335 (6)	3.4349093 (7)	6.3399610 (8)	1.1832720 (10)	2.2270714 (11)
.96	5.7351988 (6)	1.3031008 (8)	3.0066486 (9)	7.0146360 (10)	1.6503480 (12)
.97	2.4035009 (7)	7.2819128 (8)	2.2402975 (10)	6.9091688 (11)	2.1862541 (13)
.98	1.8152204 (8)	8.2499417 (9)	3.8073399 (11)	1.7766503 (13)	8.3033249 (14)
.99	5.7774227 (9)	5.2518697 (11)	4.8476752 (13)	4.5243605 (15)	4.2581255 (17)
1.00	∞	∞	∞	∞	∞

*The figures in parentheses indicate the power of ten by which the adjacent entry is to be multiplied; e.g., 1.0412490 (2) = 104.12490.

TABLE 2. *Simple power series (eq 2.1–4)*

These m-values are a lower bound on the number of terms to sum for $\Omega_j(k)$ *to* ten significant figures.

	$j=0$		$j=8$	
	$k^2=0.01$	$k^2=0.99$	$k^2=0.01$	$k^2=0.99$
Minimum m-value for which $\alpha_m(j)k^{4m}/\Omega_j(k)<10^{-11}$	3	823	4	4615

TABLE 3. *Difference sum (eq 2.2–6)*

The contribution of the difference sum to the total $\Omega_j(k)$ is here compared with the contribution from the closed-form approximation. The m-values at which difference-sum terms contribute less than a part in 10^{15} may be compared with m-values in table 2 for the simple power series.

	$j=0$		$j=8$	
	$k^2=0.01$	$k^2=0.99$	$k^2=0.01$	$k^2=0.99$
$\Omega_j(k)$ (exact)	3.141651562 (0)	5.7231768823 (0)	3.147939532 (0)	4.5243660528 (15)
Closed—form approximation:				
$\sum_{m=0}^{\infty} \alpha_m^*(j)k^{4m}$	7.071421389 (−1)	2.8256999794 (0)	3.148259078 (0)	4.5243660528 (15)
Difference—sum remainder:				
$\sum_{m=0}^{\infty} [\alpha_m(j)-\alpha_m^*(j)]k^{4m}$	2.4345094423 (0)	2.8971770330 (0)	−3.1954660509 (−4)	−4.2828876696 (−4)
Minimum m-value for which $\frac{[\alpha_m(j)-\alpha_m^*(j)]k^{4m}}{\Omega_j(k)}<10^{-11}$	3	523	2	0

TABLE 4. *Bessel function formulation for $\Omega_j(k)$ (eqs 2.3–27, 28)*

	$j=0$		$j=8$	
	$k^2=0.01$	$k^2=0.99$	$k^2=0.01$	$k^2=0.99$
$\Omega_j(k)$ − Exact	3.141 652	5.723 177	3.147 940	$4.524\ 361\times10^{+15}$
$\pi I_0\left[k^2\left(j+\frac{1}{2}\right)\right]$	3.141 612	3.337 002	3.147 270	$1.512\ 709\times10^{+3}$
$\pi I_0\left[k^2\left(j+\frac{1}{2}\right)\right]$ $-\left(\frac{3}{2}\right)k^2\left(\frac{2j+1}{6j+7}\right)I_1\left[k^2\left(j+\frac{7}{6}\right)\right]$ $+\frac{1}{4}k^4(2j+1)I_0\left[k^2\left(j+\frac{7}{6}\right)\right]$	3.141 652	3.856 095	3.147 940	$1.527\ 585\times10^{+4}$
$(3\pi/64)\left(j+\frac{1}{2}\right)\left(j+\frac{29}{18}\right)k^8$	1.2×10^{-9}	0.113 953	1.2×10^{-7}	11.556 450

TABLE 5. *Error function formulation for $\Omega_j(k)$ (eq 2.4–41)*

	$j=0$		$j=8$	
	$k^2=0.01$	$k^2=0.99$	$k^2=0.01$	$k^2=0.99$
$\Omega_j(k)$ − Exact	3.141 652	5.723 177	3.147 940	$4.524\ 361\times10^{+15}$
$\sqrt{\pi/(2j+1)}k^2(1-k^2)^{-j}$ erf λ	3.131 384	1.781 883	2.994 801	$3.822\ 991\times10^{+15}$
1st 2 terms in 2.4–41	3.144 486	3.121 919	3.177 138	$3.992\ 220\times10^{+15}$
All 3 terms in 2.4–41	3.140 552	−1.354 046	3.159 417	$3.958\ 982\times10^{+15}$

TABLE 6. *Numerical integration technique (eqs 2.5, 44–49)*

N	$j=0$		$j=8$	
	$k^2=0.01$	$k^2=0.99$	$k^2=0.01$	$k^2=0.99$
	$\Omega_j(k)$			
2	3.141 632	7.701 552	3.145 825	$52.359\ 878\times10^{+15}$
3	3.141 644	6.826 700	3.147 146	$39.269\ 908\times10^{+15}$
4	3.141 653	5.931 746	3.148 080	$24.434\ 610\times10^{+15}$
5	3.141 652	5.794 716	3.148 015	$20.725\ 792\times10^{+15}$
6	3.141 652	5.615 818	3.147 932	$15.333\ 964\times10^{+15}$
.				
.				
.				
.				
10	3.141 652	5.601 518	3.147 940	$8.430\ 212\times10^{+15}$
.				
.				
.				
.				
∞ (Exact)	3.141 652	5.723 177	3.147 940	$4.524\ 361\times10^{+15}$

Table 2, which applies to what has been called the simple power series above, illustrates how rapidly the number of terms required increases as either k or j goes up. These figures were obtained by setting the m'th term in eq. (2.1–4) equal to $\Omega_j(k)\times 10^{-11}$ and solving for m.[5] For large j and k values, the use of this formulation becomes almost prohibitively laborious.

In table 3, where the difference sum technique is examined, a number of interesting characteristics of this formulation are apparent. For small values of k, the improvement over the simple power series is negligibly small for all values of j. For k approaching unity, there is a considerable reduction in the labor involved to obtain any predetermined accuracy. For $j=0$, $k^2=0.99$ to obtain a maximum of ten significant figures, note that the number of terms required is cut from 823 to 523. While this improvement is not as spectacular as some cases which have been examined [8], it is by no means unimportant. The real gain in using this technique is seen at $j=8$, $k^2=0.99$; here we see that the closed-form approximation alone gives $\Omega_j(k)$ correct to one part in 10^{19}. This can be compared with the more than 4615 terms indicated in table 2 for ten or less significant figures using the unmodified power series.

To check the behavior of the Bessel function formulation, section 2.3 above, the BAAS tables [16] of $I_0(x)$ and $I_1(x)$ were used along with suitable interpolation tables [14]. The results are given in table 4. It will be recalled that these relations were derived essentially by approximating only the first few terms of the integrand in eq 2–1. For this reason, the formulation as would be expected yields excellent values of $\Omega_j(k)$ when $k^2=0.01$ at both $j=0$ and $j=8$; but for k^2 near unity, this technique of calculation is entirely inadequate.

The error function formulation [17], section 2.4, table 5, is also based on approximations to only the first few terms in the expansion of the integrand. Note that the leading term from 2.4–41 is not a bad approximation to the value of $\Omega_j(k)$; but the inclusion of the second or third terms results in little or no improvement. Evidently many more terms are needed to obtain useful results by this technique.

The importance of the erf and I_n formulations of $\Omega_j(k)$ arises, then, not so much for their usefulness in numerical computation, as from the relations they demonstrate between the new transcendental function considered here and older, more familiar forms.

The numerical integration formulas, from section 2.4, are quite different in kind. These, it will be observed from table 6, give excellent agreement for small k values, although they leave something to be desired for k near unity. It is interesting to note that, at these selected points, the accuracy of the formulas does not increase uniformly as N increases—for example, at $k^2=0.99$, $j=0$, the formulas with $N=5$ and 6 are virtually equivalent, and each is better than $N=10$. The approximation for large jk^2 is excellent. Note that by substituting the value of $\Omega_j(k)$ at $k^2=0.99$ and $j=8$ from table 1 it may be concluded that

$$\lim_{N\to\infty} w_N(0)\Big/\sum_{i=0}^{N} w_N(i) = \lim_{N\to\infty} \frac{(-1)^N}{N\cdot N!}\int_0^N (t-1)(t-2)(t-3)\ldots(t-N)dt \cong 0.0144\ldots$$

It thus appears that there is a multiplicity of choices available for computing $\Omega_j(k)$ for small k values, and for k approaching unity the recursion formula (1–7) can be used with the appropriate high-k expansions (1–10, 11) for ordinary elliptic integrals. However, for high-precision coverage of the entire range $0\leq k<1$ with a single formulation, only the difference-sum technique appears to be practical.

---

[5] It should be noted that, since (2.1–4) is not an alternating series, the solution for m of the inequality

$$\alpha_m(j)k^{4m}\leq\Omega_j(k)\times 10^{-11}$$

serves only as a general indication of the convergence rate and does not necessarily imply that the remainder

$$R_m(j,k)=\sum_{m+1}^{\infty}\alpha_m(j)k^{4m}$$

is less than $\Omega_j(k)\times 10^{-11}$. It does, however, indicate the *minimum* number of terms to sum, beyond which there exists, for $k^2<1$, a value $m'\geq m$ for which

$$R_{m'\geq m}(j,k)\leq\Omega_j(k)\times 10^{-11}.$$

The authors thank R. J. Herbold for programing and computing table 1, R. L. Bach for numerical checks by alternative formulations, L. C. Maximon for suggesting Stirling's formula as a basis for the difference-sum, and M. J. Berger for other helpful suggestions in the completion of this work.

4. Appendix A. Evaluation of $\alpha_m^*(j)$

Stirling's formula [11, p. 5] may be written in the form

$$z!=\sqrt{2\pi z}z^z e^{-z}H(z) \tag{A1}$$

in which $H(z)$ can be represented by the asymptotic expansion [18]

$$\lim_{z\to\infty} H(z)=1+1/(12z)+1/(288z^2)-139/(51840z^3)-\ldots. \tag{A2}$$

In order to use formula (A1) above, it is convenient to write eq (2–5) as

$$\alpha_m(j)=\frac{\pi}{(64)^m}\frac{(4m)!}{(2m)!(m!)^2}\prod_{\nu=1}^{j}\left(\frac{4m}{2\nu-1}+1\right) \tag{A3}$$

in which the product is to be taken as unity for $j=0$. Substitution of (A1) for the factorials in (A3) gives

$$\alpha_m(j)=(1/m\sqrt{2})\frac{H(4m)}{H(2m)H^2(m)}\prod_{\nu=1}^{j}\left(\frac{4m}{2\nu-1}+1\right). \tag{A4}$$

By using (A2) for the H terms, and expressing the finite product as a jth degree polynomial in m, it follows that

$$\begin{aligned}\lim_{m\to\infty}\alpha_m(j)=(1/m\sqrt{2})&\left[1-\frac{3}{16}\left(\frac{1}{m}\right)+\frac{9}{512}\left(\frac{1}{m}\right)^2+\frac{39}{8192}\left(\frac{1}{m}\right)^3+\ldots\right]\times\left[1+m\cdot 4\sum_{\nu=1}^{j}\frac{1}{2\nu-1}\right.\\&+m^2\cdot 8\sum_{\nu=1}^{j}\left\{-\frac{1}{(2\nu-1)^2}+\frac{1}{(2\nu-1)}\sum_{\mu=1}^{j}\frac{1}{2\mu-1}\right\}+\ldots\\&\left.+m^{j-2}\cdot 2^{+3j}\frac{j!}{(2j)!}\cdot\frac{1}{96}j(j-1)\,(3j^2-j-1)+m^{j-1}\cdot 2^{+3j}\frac{j!}{(2j)!}\cdot\frac{1}{4}j^2+m^j\cdot 2^{+3j}\frac{j!}{(2j)!}\right]\\&=(1/\sqrt{2})\sum_{i=0}C_i(j)m^{j-i-1}\end{aligned} \tag{A5}$$

where

$$C_0(j)=+2^{+3j}\frac{j!}{(2j)!} \tag{A6–1}$$

$$C_1(j)=+2^{+3j}\frac{j!}{(2j)!}\cdot\frac{1}{16}(4j^2-3) \tag{A6–2}$$

$$C_2(j)=+2^{+3j}\frac{j!}{(2j)!}\cdot\frac{1}{1536}(48j^4-64j^3-72j^2+16j+27) \tag{A6–3}$$

.
.
.
.

and also

$$C_j(j)=2^{-3j}\binom{2j}{j} \tag{A6–4}$$

which can be established by methods of mathematical induction. Now it is possible to show that m^n can be expressed in terms of binomial coefficients, in the form

$$m^n=\sum_{k=0}^{n}\binom{m+k}{m}\beta_{n,k} \tag{A7}$$

where the $\beta_{n,k}$'s are related to Kummer's numbers [19]. It can readily be shown that

$$\beta_{n,0}=(-1)^n \text{ and } \beta_{n,n}=n!$$

A few values of these constants are tabulated below

k \ n	0	1	2	3	4
0	+1	−1	+1	−1	+1
1	----------	+1	−3	+7	−15
2	----------	----------	+2	−12	+50
3	----------	----------	----------	+6	−60
4	----------	----------	----------	----------	+24

Now ignoring terms of the order of $1/m^2$ in the expression (A–5), for large m values, $\alpha_m(j)$ can be approximated by the expression

$$(1/\sqrt{2})\left[\sum_{i=0}^{j-1}C_i(j)m^{j-i-1}+C_j(j)m^{-1}\right].$$

Replacing $1/m$ by its equivalent to this order, $1/(m+1)$, and using (A–7) in the sum term, it follows that

$$\alpha_m^*(j)=(1/\sqrt{2})\left[\sum_{i=0}^{j-1}C_i(j)\sum_{k=0}^{j-i-1}\beta_{j-i-1,k}\binom{m+k}{m}+C_j(j)\frac{1}{m+1}\right]$$

$$=(1/\sqrt{2})2^{-3j}\binom{2j}{j}\left\{\frac{1}{m+1}+\sum_{i\geq 0}^{j-1}B_{ij}\binom{m+j-i-1}{m}\right\} \tag{A–8}$$

which is eq 2.2–8 of the text above. The B_{ij} expression in eq (A–8) is obtained by combining the $C_i(j)$ and $\beta_{n,k}$ values as indicated and factoring out $C_j(j)$. The expression (2.2–11) above is then obtained using the familiar series

$$\ln(1-x)=-\sum_{m=0}^{\infty}x^{m+1}/(m+1) \tag{A9}$$

and

$$(1-x)^{-k-1}=\sum_{m=0}^{\infty}\binom{m+k}{m}x^m. \tag{A10}$$

5. References

[1] J. H. Hubbell, R. L. Bach, and R. J. Herbold, Radiation field from a circular disk source, J. Research NBS **65C** (Engr. and Instr.) No. 4, 249 (1961). In this paper, the notation $S_n(k)$ is used for the function which here is designated as $\Omega_j(k)$.

[2] M. J. Berger and J. C. Lamkin, Sample calculations of gamma-ray penetration into shelters: Contributions of sky shine and roof contamination, J. Research NBS **60,** 109 (1958) RP2827.
[3] P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists (Springer, Berlin, 1954).
[4] B. O. Peirce, A Short Table of Integrals, 4th Ed., revised by R. M. Foster, (Ginn, New York, 1957).
[5] H. B. Dwight, Tables of Integrals and Other Mathematical Data, 3rd Ed. (MacMillan, New York, 1957).
[6] A. Fletcher, Guide to tables of elliptic functions, M.T.A.C. **3,** 229 (1948).
[7] C. Hastings, Jr., Approximations for Digital Computers (University Press, Princeton, 1955).
[8] L. F. Epstein and N. E. French, Improving the convergence of series: Application to some elliptic integrals, Am. Math. Monthly **63,** 698 (1956).
[9] E. P. Adams, Ed., Smithsonian Mathematical Formulae and Tables of Elliptic Functions (Smithsonian Institution, Washington, 1957), 3d Reprint.
[10] C. Jordan, Calculus of Finite Differences (Chelsea Publ. Co., New York, 1947) 2d ed., p. 69.
[11] E. Jahnke, and F. Emde, Tables of Higher Functions, 6th Edition, Friedrich Losch, Ed. (McGraw-Hill, New York, 1960).
[12] G. N. Watson, Theory of Bessel Functions, 2nd Ed. (University Press, Cambridge, 1944).
[13] K. Pearson, Tables of the Incomplete Gamma Function (University Press, Cambridge, 1957).
[14] Mathematical Tables Project, WPA, Tables of Lagrangian Interpolation Coefficients. 2d Printing. (Columbia University Press, New York, 1948).
[15] E. T. Whittaker and G. Robinson, The Calculus of Observations, 4th Ed. (Blackie, London, 1958).
[16] British Association for the Advancement of Science, Mathematical Tables, Volume VI. Bessel Functions, Part I. (University Press, Cambridge, 1958).
[17] National Bureau of Standards, Tables of the error function and its derivatives, Applied Mathematics Series, No. 41 (Washington, 1954).
[18] For coefficients of this expansion through the $1/z^7$ terms, see H. T. Davis, Tables of the Higher Mathematical Functions, Volume I, p. 180. (Principia Press, Bloomington, Indiana, 1933.)
[19] E. E. Kummer, J. F. Math. (Crelle) **21,** 74 (1840), See also Fletcher, A., Miller, J. C. P., Rosenhead, L. and Comrie, L. J.: An Index of Mathematical Tables, 2d Ed. (Addison-Wesley, Reading, Mass., 1962) vol. I, p. 108.

(Paper 67B1–88)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

An Algorithm for Obtaining an Orthogonal Set of Individual Degrees of Freedom for Error

Joseph M. Cameron

(November 20, 1962)

This note presents an algorithm based on the Gram-Schmidt orthonormalization procedure for producing the coefficients of linear combinations of observations which can be used for computing an orthogonal set of individual degrees of freedom for error from a set of observations.

In the analysis of data from designed experiments it is becoming common to compute the deviations between the observed and predicted values. Because of the correlation among these residuals it is sometimes easier to interpret a set of independent individual degrees of freedom, particularly if the residuals are to be used to study the state of statistical control of a measurement process. This note presents an algorithm for producing such an orthogonal set.

Let the observations $Y_1, Y_2, \ldots Y_n$ have expected values

$$\begin{aligned} E(Y_1) &= \beta_0 X_{01} + \beta_1 X_{11} + \ldots + \beta_k X_{k1} \\ E(Y_2) &= \beta_0 X_{02} + \beta_1 X_{12} + \ldots + \beta_k X_{k2} \\ &\ldots \\ E(Y_n) &= \beta_0 X_{0n} + \beta_1 X_{1n} + \ldots + \beta_k X_{kn} \end{aligned}$$

which can be written in matrix notation as

$$E(y) = X\beta$$

where y is the vector of observations, X the array of X_{ij}'s and β a vector of parameters.

The deviation, δ_i, between the observed and predicted value is then

$$\delta_i = y_i - \hat{\beta}_0 X_{0i} - \hat{\beta}_1 X_{1i} - \ldots - \hat{\beta}_k X_{ki}$$

or in matrix form

$$\delta = y - X\hat{\beta}$$

where $\hat{\beta}$ is the vector of estimates of the parameters of the system. Noting that the $\hat{\beta}_i$ are functions of the observations each δ_i can also be written as a function of the observation, say

$$\delta = Ay.$$

To say that there are m degrees of freedom for error is to say that A has rank m or that the rows of A are linear combinations of a set of m orthogonal rows. These m orthogonal rows give the coefficients necessary for computing the individual degrees of freedom for error.

The orthogonalization can be carried out by an application of the Gram-Schmidt process,[1]

[1] Philip Davis and Philip Rabinowitz, A multiple purpose orthonormalizing code and its uses, Journal of the Association for Comp uting Machinery **1**, 183-191 (1954).

a modification of which follows. The deviations can be written as function of the observations as follows:

$$\begin{aligned}
\delta_1 &= a_{11}y_1 + a_{12}y_2 + \ \ldots\ + a_{1n}y_n \\
\delta_2 &= a_{21}y_1 + a_{22}y_2 + \ \ldots\ + a_{2n}y_n \\
&\ldots \\
\delta_n &= a_{n1}y_1 + a_{n2}y_2 + \ \ldots\ + a_{nn}y_n.
\end{aligned}$$

Consider now the vectors

$$\begin{aligned}
a_1 &= (a_{11},\ a_{12},\ \ldots,\ a_{1n}) \\
a_2 &= (a_{21},\ a_{22},\ \ldots,\ a_{2n}) \\
&\ldots \\
a_n &= (a_{n1},\ a_{n2},\ \ldots,\ a_{nn})
\end{aligned}$$

and denote the sums of squares and cross products as follows

$$(a_k \cdot a_k) = \sum_{i=1}^{n} a_{ki}^2$$

$$(a_k \cdot a_l) = \sum_{i=1}^{n} a_{ki}a_{li}.$$

Form the vectors

$$\begin{aligned}
b_1 &= a_1 \\
b_2 &= (a_1 \cdot a_1)a_2 - (a_1 \cdot a_2)a_1 \\
b_3 &= (a_1 \cdot a_1)a_3 - (a_1 \cdot a_3)a_1 \\
&\ldots \\
b_n &= (a_1 \cdot a_1)a_n - (a_1 \cdot a_n)a_1
\end{aligned}$$

where the vectors b_i are the vector sums of the indicated vectors, e.g., the elements of b_2 are

$$(a_1 \cdot a_1)a_{21} - (a_1 \cdot a_2)a_{11},\ (a_1 \cdot a_1)a_{22} - (a_1 \cdot a_2)a_{12},\ \ldots,\ (a_1 \cdot a_1)a_{2n} - (a_1 \cdot a_2)a_{1n}.$$

Next form

$$\begin{aligned}
c_1 &= b_1 \\
c_2 &= b_2 \\
c_3 &= (b_2 \cdot b_2)b_3 - (b_2 \cdot b_3)b_2 \\
c_4 &= (b_2 \cdot b_2)b_4 - (b_2 \cdot b_4)b_2 \\
&\ldots \\
c_n &= (b_2 \cdot b_2)b_n - (b_2 \cdot b_n)b_2
\end{aligned}$$

Then

$$\begin{aligned}
d_1 &= c_1 \\
d_2 &= c_2 \\
d_3 &= c_3 \\
d_4 &= (c_3 \cdot c_3)c_4 - (c_3 \cdot c_4)c_3 \\
d_5 &= (c_3 \cdot c_3)c_5 - (c_3 \cdot c_5)c_3 \\
&\ldots
\end{aligned}$$

If in the process one of the vectors becomes all zeros, it should be transferred to the end and the vectors renumbered accordingly.

This process is continued for m steps at which point the last n-m vectors will be zero vectors and the first m will be the desired orthogonal set. It is easy to verify that at the kth stage the first $(k-1)$ vectors are mutually orthogonal and orthogonal to all vectors below.

To illustrate, consider the design where all differences among four objects are measured i.e.,

$$
\begin{aligned}
E(Y_1) &= \beta_1 - \beta_2 \\
E(Y_2) &= \beta_1 - \beta_3 \\
E(Y_3) &= \beta_1 - \beta_4 \\
E(Y_4) &= \beta_2 - \beta_3 \\
E(Y_5) &= \beta_2 - \beta_4 \\
E(Y_6) &= \beta_3 - \beta_4 .
\end{aligned}
$$

The following table shows the estimates of the β's under the assumption that $\Sigma\beta = 0$.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
$4\hat{\beta}_1$	1	1	1	0	0	0
$4\hat{\beta}_2$	−1	0	0	1	1	0
$4\hat{\beta}_3$	0	−1	0	−1	0	1
$4\hat{\beta}_4$	0	0	−1	0	−1	−1
$4\delta_1 = 4Y_1 - 4(\hat{\beta}_1 - \hat{\beta}_2)$	2	−1	−1	1	1	0
$4\delta_2 = 4Y_2 - 4(\hat{\beta}_1 - \hat{\beta}_3)$	−1	2	−1	−1	0	1
$4\delta_3 = 4Y_3 - 4(\hat{\beta}_1 - \hat{\beta}_4)$	−1	−1	2	0	−1	−1
$4\delta_4 = 4Y_4 - 4(\hat{\beta}_2 - \hat{\beta}_3)$	1	−1	0	2	−1	1
$4\delta_5 = 4Y_5 - 4(\hat{\beta}_2 - \hat{\beta}_4)$	1	0	−1	−1	2	−1
$4\delta_6 = 4Y_6 - 4(\hat{\beta}_3 - \hat{\beta}_4)$	0	1	−1	1	−1	2

From this table one has, for example:

$$
\begin{aligned}
\hat{\beta}_1 &= \tfrac{1}{4}(Y_1 + Y_2 + Y_3) \\
\delta_1 &= \tfrac{1}{4}(2Y_1 - Y_2 - Y_3 + Y_4 + Y_5)
\end{aligned}
$$

The matrix A is then

$$
\begin{aligned}
a_1 &= (\ 2 \quad -1 \quad -1 \quad 1 \quad 1 \quad 0) & (a_1a_1) &= 8 \\
a_2 &= (-1 \quad 2 \quad -1 \quad -1 \quad 0 \quad 1) & (a_1a_2) &= -4 \\
a_3 &= (-1 \quad -1 \quad 2 \quad 0 \quad -1 \quad -1) & (a_1a_3) &= -4 \\
a_4 &= (\ 1 \quad -1 \quad 0 \quad 2 \quad -1 \quad 1) & (a_1a_4) &= 4 \\
a_5 &= (\ 1 \quad 0 \quad -1 \quad -1 \quad 2 \quad -1) & (a_1a_5) &= 4 \\
a_6 &= (\ 0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 2) & (a_1a_6) &= 0
\end{aligned}
$$

The first step gives

$$
\begin{aligned}
b_1 &= a_1 = (2 \quad -1 \quad -1 \quad 1 \quad 1 \quad 0) & & \\
b_2 &= \tfrac{1}{4}(8a_2 + 4a_1) = (0 \quad 3 \quad -3 \quad -1 \quad 1 \quad 2) & (b_2b_2) &= 24 \\
b_3 &= \tfrac{1}{4}(8a_3 + 4a_1) = (0 \quad -3 \quad 3 \quad 1 \quad -1 \quad -2) & (b_2b_3) &= -24 \\
b_4 &= \tfrac{1}{4}(8a_4 - 4a_1) = (0 \quad -1 \quad 1 \quad 3 \quad -3 \quad 2) & (b_2b_4) &= -8 \\
b_5 &= \tfrac{1}{4}(8a_5 - 4a_1) = (0 \quad 1 \quad -1 \quad -3 \quad 3 \quad -2) & (b_2b_5) &= 8 \\
b_6 &= \tfrac{1}{8}(8a_6 \quad\ ) = (0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 2) & (b_2b_6) &= 8.
\end{aligned}
$$

The vectors b_2, b_3, . . . are multiplied by factors (e.g., 1/4, 1/4, . . .) to keep the numbers conveniently small.

The second step gives

$$
\begin{array}{rrrrrrr}
c_1 = & & b_1 = (2 & -1 & -1 & 1 & 1 & 0) \\
c_2 = & & b_2 = (0 & 3 & -3 & -1 & 1 & 2) \\
c_3 = & 24b_3 + 24b_2 = (0 & 0 & 0 & 0 & 0 & 0) \\
c_4 = & \tfrac{1}{84}(24b_4 + 8b_2) = (0 & 0 & 0 & 1 & -1 & 1) \\
c_5 = & \tfrac{1}{84}(24b_5 - 8b_2) = (0 & 0 & 0 & -1 & 1 & -1) \\
c_6 = & \tfrac{1}{32}(24b_6 - 8b_2) = (0 & 0 & 0 & 1 & -1 & 1)
\end{array}
$$

Vector c_3 is transferred to the end to give

$$
\begin{array}{rrrrrrr}
c_1 = (2 & -1 & -1 & 1 & 1 & 0) & \\
c_2 = (0 & 3 & -3 & -1 & 1 & 2) & \\
c_3 = (0 & 0 & 0 & 1 & -1 & 1) & (c_3c_3) = 3 \\
c_4 = (0 & 0 & 0 & -1 & 1 & -1) & (c_3c_4) = -3 \\
c_5 = (0 & 0 & 0 & 1 & -1 & 1) & (c_3c_5) = 3 \\
c_6 = (0 & 0 & 0 & 0 & 0 & 0) & (c_3c_6) = 0
\end{array}
$$

The next stage gives

$$
\begin{array}{rrrrrrr}
d_1 = & c_1 = (2 & -1 & -1 & 1 & 1 & 0) \\
d_2 = & c_2 = (0 & 3 & -3 & -1 & 1 & 2) \\
d_3 = & c_3 = (0 & 0 & 0 & 1 & -1 & 1) \\
d_4 = 3c_4 + 3c_3 = (0 & 0 & 0 & 0 & 0 & 0) \\
d_5 = 3c_5 - 3c_3 = (0 & 0 & 0 & 0 & 0 & 0) \\
d_6 = & 0 = (0 & 0 & 0 & 0 & 0 & 0)
\end{array}
$$

with d_1, d_2, d_3, being the desired set of orthogonal vectors.

It is worth noting that one need not start with the deviations, but may begin with any set of linear functions having zero expectation and rank equal to the number of degrees of freedom for error. For the example the three vectors below could have been used.

$$
\begin{array}{rrrrrrr}
a_1 = (1 & -1 & 0 & 1 & 0 & 0) \\
a_2 = (0 & 0 & 0 & 1 & -1 & 1) \\
a_3 = (0 & 1 & -1 & 0 & 0 & 1)
\end{array}
$$

The steps in the orthogonalization are

$$
\begin{array}{rrrrrrr}
b_1 = & a_1 = ; & 1 & -1 & 0 & 1 & 0 & 0) \\
b_2 = 3a_2 - a_1 = (-1 & & 1 & 0 & 2 & -3 & 3) \\
b_3 = 3a_3 + a_1 = (1 & & 2 & -3 & 1 & 0 & 3)
\end{array}
$$

$$
\begin{array}{rrrrrrr}
c_1 = & b_1 = (1 & -1 & 0 & 1 & 0 & 0) \\
c_2 = & b_2 = (-1 & 1 & 0 & 2 & -3 & 3) \\
c_3 = \tfrac{1}{36}(24b_3 - 12b_2) = (1 & 1 & -2 & 0 & 1 & 1).
\end{array}
$$

(Paper 67B1–89)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

Recognition of Completely Mixed Games

A. J. Goldman

(November 14, 1962)

A matrix game is called completely mixed if no optimal strategy has a zero component. J. von Neumann's necessary and sufficient "separated diagonals" condition for a 2-by-2 game to be completely mixed was extended by Bohnenblust, Karlin, and Shapley to a sufficient condition for a general matrix game to be completely mixed. The present paper gives still weaker sufficient conditions, thus facilitating recognition of a wider class of completely mixed games as such. Special stress is put on the possibility of using row and column permutations to transform a given matrix into one obeying the conditions.

1. Introduction

Let $A=(a_{ij})$ be an m-by-n matrix with $i=0,1,\ldots,m-1$ and $j=0,1,\ldots,n-1$. We regard A as the *payoff table* of a finite zero-sum two-person game; if Player I chooses his ith course of action and Player II chooses his jth course of action, then the outcome of the game is such that the rules prescribe a payment of a_{ij} units to Player I by Player II. (If $a_{ij}<0$, the "physical" payment would go in the opposite direction.)

An m-component row vector $X=(x_0,\ldots,x_{m-1})$ such that

$$\sum_{i=0}^{m-1} x_i=1, \qquad \text{all } x_i\geq 0$$

can be interpreted as a *mixed strategy* (i.e., a probabilistic mixture of courses of action) for Player I, with x_i representing the relative frequency or probability with which he employs his ith course of action. Similarly, an n-component column vector Y with nonnegative components y_j summing to unity can be interpreted as a mixed strategy for Player II. The bilinear form XAY then represents the expected value of the payoff to Player I by Player II, if they select the respective mixed strategies X and Y.

A triple (v,X^*,Y^*), where v is a number and X^* and Y^* are mixed strategies for Players I and II respectively, is called a *solution* of the game if

$$XAY^*\leq v\leq X^*AY$$

holds for all mixed strategies X and Y. The celebrated *minimax theorem* of J. von Neumann asserts the existence of at least one solution. The number v is called the *value* of the game; though equal to X^*AY^*, it can be shown to be independent of the particular solution. A mixed strategy is called *optimal* if it appears in at least one solution, and it is known that a mixed strategy X^* is optimal if and only if

$$\sum_{i=0}^{m-1} x_i^* a_{ij}\geq v \qquad \text{for } j=0,1,\ldots,n-1, \tag{1}$$

while a mixed strategy Y^* is optimal if and only if

$$\sum_{j=0}^{n-1} a_{ij} y_j^*\leq v \qquad \text{for } i=0,1,\ldots,m-1. \tag{2}$$

We will make use of the easily proved fact that if strict inequality holds in (1) for some particular j and *any* optimal X^*, then $y_j^*=0$ for *every* optimal Y^*, while if strict inequality holds in (2) for some particular i and *any* optimal Y^*, then $x_i^*=0$ for *every* optimal X^*.

The game is called *completely mixed* if every optimal strategy, for each player, has all its components strictly positive. Intuitively this means that depriving a player of one of his courses of action would really damage him versus a rational opponent. It is known [1,2] that a completely mixed game must have a *square* matrix, and so $m=n$ will be assumed in what follows.

For $0\leq k\leq n-1$ let D_k denote the k-th *diagonal* of A, the set of entries a_{ij} with $j-i=k \pmod n$. A "classical" result of von Neumann [3] asserts that a 2-by-2 game is completely mixed if and only if its diagonals are *separated*, i.e., if there are disjoint intervals I_0 and I_1 with $D_0\subset I_0$ and $D_1\subset I_1$. A generalization was given by Bohnenblust, Karlin, and Shapley,[4] who showed that an n-by-n game is completely mixed if its diagonals are both separated and ordered, i.e., if there are disjoint intervals in the order

[1] I. Kaplansky, A contribution to von Neumann's theory of games, Annals of Math. **46** (1945).
[2] D. Gale and S. Sherman, Solutions of finite two-person games, Paper 4 of Princeton Annals of Math. Study No. 24 (1953), Contributions to the Theory of Games. See p. 48.
[3] O. Morgenstern and J. von Neumann, Theory of Games and Economic Behavior, Princeton University Press (1947). See p. 173.
[4] H. F. Bohnenblust, S. Karlin, and L. S. Shapley, Solutions of discrete two-person games, Paper 5 of Princeton Annals of Math. Study No. 24 (1950), Contributions to the Theory of Games. See pt III.

$$I_0 \subset I_1 \subset \ldots \subset I_{n-1} \qquad (3)$$

such that $D_k \subset I_k$ for $0 \leq k \leq n-1$. They point out, however, that this sufficient condition is not *necessary* for complete mixture even when $n=3$.

It can be shown (*op. cit.* in footnote 4) that a game with payoff matrix A is completely mixed if and only if the same is true of all games whose matrices are obtainable from A by a sequence of operations each of one of the following types:

(i) Permutation of rows.
(ii) Permutation of columns.
(iii) Matrix transposition.
(iv) Matrix negation.

The diagonals-separated-and-ordered condition (briefly, the "BKS condition") for complete mixture may apply to one of these transforms of A but not to A itself, so that *a priori* the criterion can only be used "fully" by testing the diagonals of *each* transform. It is fairly obvious that actual generation of all the transforms of A can be replaced by a more efficient procedure, and theorem 1 of section 2 can be viewed as the rather straightforward justification of one such procedure, or alternatively as a determination of just how far the range of applicability of the BKS theorem is extended by the operations (i) through (iv).

Theorem 2 in section 3 gives a new sufficient condition for complete mixture. Though strictly weaker than the BKS condition (entries of A in the same diagonal are not all lumped together), it is in the same general spirit, permitting a wider class of completely mixed games to be recognized as such "by inspection" if this term is generously interpreted. The ideas of section 2 are carried over to this new context, and theorem 3 deals with the possibility of transforming a given matrix into one obeying the sufficient condition of theorem 2.

Theorems 1, 2, and 3 are the main results of the paper. Three related topics are treated in section 4. First, a sufficient condition for complete mixture apparently still weaker than that of theorem 2 is given (theorem 4), but is shown (in theorem 5) to be equivalent when both conditions are aided by the transformations (i) through (iv). Second, it is shown by example that our results are not implied by a second sufficient condition for complete mixture due to Bohnenblust, Karlin, and Shapley. Third, it is noted that our criteria still fail to identify all 3×3 completely mixed games, in part because they apply only when each row and each column of the payoff matrix consists of distinct entries. Two results are given which require only "one-way" distinctness (theorem 6 for columns, theorem 7 for rows), but they do not resolve the case $n=3$.

2. Testing the Applicability of the BKS Condition

First a simplification will be made. Suppose a sequence of operations, of the types (i) through (iv) listed in the introduction, leads from a matrix A to a matrix A' and contains exactly t matrix transpositions and exactly N matrix negations. Both transposition and negation are involutions, i.e., $(B^T)^T=B$ and $-(-B)=B$. Also, a row (column) permutation followed by a transposition is equivalent to the same permutation applied to the columns (rows) *preceded* by a transposition, while any row or column permutation commutes with negation. From these remarks it follows that A' can be obtained by applying an appropriate sequence, consisting of row and column permutations *only*, to

$$\begin{array}{rl} A & \text{for } t \text{ even, } N \text{ even,} \\ -A & \text{for } t \text{ even, } N \text{ odd,} \\ A^T & \text{for } t \text{ odd, } N \text{ even,} \\ -A^T & \text{for } t \text{ odd, } N \text{ odd.} \end{array}$$

We shall therefore delete transposition and negation from the repertoire of allowable operations; a *transform* of A now will mean a matrix obtainable from A by a sequence of row and/or column permutations. If C denotes a necessary and sufficient condition that at least one transform of A has some property, then the statement

"A or $(-A)$ or A^T or $(-A^T)$ satisfies C"

is a necessary and sufficient condition that at least one matrix obtainable from A by *all* the operations (i) through (iv) of the introduction has the property.

Next some terminology and notation will be introduced. A pair of matrix entries[5] is called *collinear* if the entries lie in the same row or the same column. A set of n entries (in an n-by-n matrix $A=(a_{ij})$) will be called a *chord* if it contains no collinear pair; the chords of A are precisely the sets of the form

$$S(A,\pi)=\{a_{i,\pi(i)}:\ i=0,1,\ldots,n-1\},$$

where π is a permutation of $\{0,1,\ldots,n-1\}$, determined uniquely by the chord. For example, the diagonals of A are the sets

$$D_k(A)=S(A,\delta_k) \qquad k=0,1,\ldots,n-1$$

associated with the "diagonal permutations" δ_k defined by

$$\delta_k(i)=i+k \pmod{n}.$$

Suppose $A'=(a'_{ij})$ is obtained from $A=(a_{ij})$ by a row permutation σ and a column permutation τ, so that $a_{ij}=a'_{\sigma(i),\tau(j)}$. Setting $t=\sigma(i)$ leads to

$$a_{i,\,\pi(i)}=a'_{t,\tau\pi\sigma^{-1}(t)}\epsilon S(A',\tau\pi\sigma^{-1}),$$

and it is easily shown by such reasoning that

$$S(A,\pi)=S(A',\tau\pi\sigma^{-1}) \qquad (4)$$

[5] To avoid cumbersome notation, we occasionally slur the distinction between a matrix entry a_{ij} and its position (i,j); this should cause no confusion.

for any permutation π. If A' is obtained from A by a sequence consisting of row permutations $\sigma_1, \sigma_2, \ldots, \sigma_p$ and column permutations $\tau_1, \tau_2, \ldots, \tau_q$, each listed in their order of appearance in the sequence, then eq (4) still holds with the definitions[6]

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_p, \qquad \tau = \tau_1 \tau_2 \ldots \tau_q.$$

This fact will be used in the following proof.

THEOREM 1: *Some transform of* A *has separated and ordered diagonals, if and only if*

(i) *the sets* $S_k = \{a_r: kn \leq r < (k+1)n\}$ *determined by an enumeration* $\{a_r: 0 \leq r < n^2\}$ *of the entries of* A *in nondecreasing order are independent of the particular enumeration,*[7]

(ii) *each* S_k *is a chord, and thus determines a permutation* π_k *such that* $S_k = S(A, \pi_k)$, *and*

(iii) $\pi_k \pi_0^{-1} = (\pi_1 \pi_0^{-1})^k$ *for* $k = 0, 1, \ldots, n-1$.

The necessity of the conditions will be proved first. Suppose A' is a transform of A which has separated and ordered diagonals. The set of entries of A' (multiplicities included) is the same as for A, so that

$$S_k = D_k(A') \qquad \text{for } k = 0, 1, \ldots, n-1.$$

This shows that the sets S_k are unambiguously defined. Since row and column permutations preserve collinearity of pairs and therefore map only chords into chords, and since $D_k(A')$ is a chord of A', its "pretransform" S_k must be a chord of A. Thus only the necessity of (iii) remains to be verified.

Since A' is a transform of A, there are permutations σ and τ such that eq (4) holds for all π. In particular, because

$$S(A, \pi_k) = S_k = D_k(A') = S(A', \delta_k),$$

it follows that

$$\delta_k = \tau \pi_k \sigma^{-1} \qquad \text{for } k = 0, 1, \ldots, n-1. \qquad (5)$$

Since δ_0 is the identity permutation, this equation yields $\sigma = \tau \pi_0$, and therefore implies

$$\pi_k \pi_0^{-1} = \tau^{-1} \delta_k \tau = (\tau^{-1} \delta_1 \tau)^k \qquad \text{for } k = 0, 1, \ldots, n-1$$

so that (iii) holds.

For the sufficiency proof, suppose the conditions hold. Define a mapping τ^{-1} of $\{0, 1, \ldots, n-1\}$ into itself by

$$\tau^{-1}(k) = \pi_k \pi_0^{-1}(0) = (\pi_1 \pi_0^{-1})^k(0).$$

To see that τ^{-1} is a permutation (so that τ is well-defined), it suffices to observe that by their definitions π_i and π_k, for $i \neq k$, can agree for no value of the "independent variable" (i.e., S_i and S_k are disjoint) and thus not at $\pi_0^{-1}(0)$. Now define $\sigma = \tau \pi_0$. Then by (iii)

$$\tau \pi_k \sigma^{-1} = \tau(\pi_k \pi_0^{-1}) \tau^{-1} = \tau(\pi_1 \pi_0^{-1})^k \tau^{-1},$$

so that

$$\tau \pi_k \sigma^{-1}(i) = \tau(\pi_1 \pi_0^{-1})^k \tau^{-1}(i) = \tau(\pi_1 \pi_0^{-1})^k (\pi_1 \pi_0^{-1})^i(0)$$
$$= (\pi_1 \pi_0^{-1})^{i+k}(0) \equiv i + k = \delta_k(i) \pmod{n}. \qquad (6)$$

Thus eq (5) holds, and so the transform of A by row permutation σ and column permutation τ has separated and ordered diagonals.

From the first paragraph of section 2 it would seem appropriate to apply the criterion of theorem 1 to all four of the matrices $\pm A$, $\pm A^T$. Fortunately this is unnecessary; the criterion can be applied to A alone without loss of information. That is, the four matrices all obey the criterion if and only if any one of them does. In checking this assertion, the equivalence of the criterion for A and $(-A^T)$ follows by "composition" from its equivalence for A and A^T, and for A and $(-A)$. Since matrix negation and transposition are involutions, it suffices to prove that if A obeys the three conditions of theorem 1, then the same is true of $(-A)$ and A^T. The routine but tedious verification is omitted.

We conclude this section with some examples:

Example 1.
$$A_1 = \begin{pmatrix} 2 & 5 & 6 \\ 4 & -1 & 1 \\ 3 & 7 & 0 \end{pmatrix}.$$

Here $S_0 = \{-1, 0, 1\}$ and $S_2 = \{5, 6, 7\}$ are not chords, so that condition (ii) fails and the BKS theorem (aided by operations (i) through (iv) of the introduction) does not identify A_1 as the payoff matrix of a completely mixed game.

Example 2.
$$A_2 = \begin{pmatrix} 4 & -3 & -2 \\ -3 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here condition (i) fails, for either appearance of (-2) could be assigned to S_0 and the other to S_1 (see footnote 7); again the BKS theorem is inapplicable.

Example 3.
$$A_3 = \begin{pmatrix} 3 & 0 & 2 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

[6] Our convention is that the factors in a product of permutations operate in right-to-left order. If $p = 0$ we take σ to be the identity permutation; if $q = 0$ we take τ to be the identity permutation.

[7] I.e., the smallest element of S_{k+1} strictly exceeds the largest element of S_k, for $k = 0, 1, \ldots, n-1$. This is certainly true if all entries of A are distinct, but is also true (for example) if A has n distinct elements, each appearing in n positions.

Here conditions (i) and (ii) are satisfied; S_k consists of four appearances of k. We represent a permutation by writing each value of its "independent variable" above the corresponding value of the "dependent variable." Thus

$$\pi_0=\begin{pmatrix}0&1&2&3\\1&3&0&2\end{pmatrix} \qquad \pi_1=\begin{pmatrix}0&1&2&3\\3&2&1&0\end{pmatrix}$$

$$\pi_2=\begin{pmatrix}0&1&2&3\\2&0&3&1\end{pmatrix} \qquad \pi_3=\begin{pmatrix}0&1&2&3\\0&1&2&3\end{pmatrix}.$$

We leave it to the reader to verify that all three conditions of theorem 1 are satisfied, and that using the column permutation τ defined by

$$\tau^{-1}=\begin{pmatrix}0&1&2&3\\0&1&3&2\end{pmatrix}$$

and the row permutation

$$\sigma=\tau\pi_0=\begin{pmatrix}0&1&2&3\\1&2&0&3\end{pmatrix},$$

as in the sufficiency proof of theorem 1, leads to

$$A_3'=\begin{pmatrix}0&1&2&3\\3&0&1&2\\2&3&0&1\\1&2&3&0\end{pmatrix},$$

a transform of A with separated and ordered diagonals.

3. Extensions of the BKS theorem

The previous section was essentially combinatorial; the present one, in contrast, has some specifically game-theoretic content. We begin with the following simple generalization of the BKS theorem:[8]

THEOREM 2: *The game with* n-*by*-n *matrix* A *is completely mixed if there exist integers* p,q,r,s *with sum relatively prime to* n, *such that*

(a) $a_{ij}<a_{i,j+p}$ *whenever* $j-i\not\equiv q \pmod n$,

(b) $a_{ij}>a_{i+r,j}$ *whenever* $i-j\not\equiv s \pmod n$.

Before proving this result, we observe that it implies the BKS theorem; if A has separated and ordered diagonals, then theorem 2 applies with $p=r=1$, $q=n-1$, and $s=0$. Theorem 2 is *strictly* stronger than the BKS theorem, for it applies (also with $p=r=1$, $q=n-1$, and $s=0$) to show that the matrix A_1 of example 1 represents a completely mixed game, a fact which the BKS theorem (even aided by row and column, permutations, matrix transposition, and negation) could not reveal, as was demonstrated in section 2.

For the proof, suppose that some component x_k^* of an optimal strategy X^* vanished. We will show that this implies

$$x^*_{k+p+q+r+s}=0. \tag{7}$$

The same argument can then be applied with $k+p+q+r+s$ replacing k. Repetition of the argument, together with the hypothesis on $p+q+r+s$, shows that *all* components of X^* vanish, which is absurd since these components must sum to unity. Thus no component of any optimal X^* can vanish; a similar analysis applies for optimal Y^*, and so the game is completely mixed.

To prove eq (7), first use (1) and the optimality of X^* to write

$$v\le\sum_{i=0}^{n-1}x_i^*a_{i,k+q}=\sum_{i\neq k}x_i^*a_{i,k+q}.$$

By hypothesis (a) of the theorem we have $a_{i,k+q}<a_{i,k+p+q}$ for each term in the last sum, and at least one of the x_i^* in this sum is positive (they are all nonnegative). Therefore

$$v<\sum_{i\neq k}x_i^*a_{i,k+p+q}=\sum_{i=0}^{n-1}x_i^*a_{i,k+p+q}.$$

By the remark immediately following (1) and (2), y^*_{k+p+q} vanishes for every optimal Y^*, so that by (2)

$$v\ge\sum_{j=0}^{n-1}a_{k+p+q+s,\,j}y_j^*=\sum_{j\neq k+p+q}a_{k+p+q+s,\,j}y_j^*.$$

By hypothesis (b) of the theorem we have $a_{k+p+q+s,\,j}>a_{k+p+q+r+s,\,j}$ for each term in the last sum, and at least one of the y_j^* in this sum is positive (they are all nonnegative). Therefore

$$v>\sum_{j\neq k+p+q}a_{k+p+q+r+s,\,j}y_j^*=\sum_{j=0}^{n-1}a_{k+p+q+r+s,\,j}y_j^*,$$

which inplies eq (7) by the comment following (1) and (2). This completes the proof of the theorem.

To derive full benefit from theorem 2, we should determine its range of applicability when aided by the four operations listed in the introduction. These aims are accomplished in the next theorem, which is related to theorem 2 as theorem 1 is related to the BKS theorem. First two lemmas will be given.

LEMMA 1. *Suppose that each row and each column of* A *has a unique minimum entry, that the row minima form a chord, and that the column minima form a chord. Then the row and column minima coincide.*

For the proof, suppose for example that $a_{i(0),j(0)}$ is a row minimum but not a column minimum. Let $a_{i(1),j(0)}$, with $i(1)\neq i(0)$, be the minimum of column $j(0)$. It is not the minimum of row $i(1)$, since column $j(0)$ contains only one row minimum (the one in row $i(0)$). Let $a_{i(1),j(1)}$, with $j(1)\neq j(0)$, be

[8] In what follows, appropriate numbers should be read modulo n.

the minimum of row i(1). It is not the minimum of column j(1), etc. Continuing similarly, we obtain an infinite sequence of entries of A, which is strictly decreasing and therefore nonrepeating. Since this is impossible, every row minimum $a_{i(0),j(0)}$ must also be a column minimum.

LEMMA 2. *Suppose that each line*[9] *of* A *has distinct entries, so that the sets* $\mathrm{R_k}$ *of* k-th *smallest row entries*[10] *and* $\mathrm{C_k}$ *of* k-th *smallest column entries are uniquely defined for* $0 \leq \mathrm{k} < \mathrm{n}$. *If each* $\mathrm{R_k}$ *and each* $\mathrm{C_k}$ *is a chord, then* $\mathrm{R_k} = \mathrm{C_k}$ *for all* k.

For, $R_0 = C_0$ by lemma 1. Now replace all entries in $R_0 = C_0$ by numbers greater than any other entries of A. The result is a matrix A^* for which (with an obvious notation) $R_{k-1}(A^*) = R_k(A)$ and $C_{k-1}(A^*) = C_k(A)$ for $1 \leq k \leq n-1$, while $R_{n-1}(A^*) = C_{n-1}(A^*)$ consists of the new entries. By lemma 1,

$$R_1(A) = R_0(A^*) = C_0(A^*) = C_1(A),$$

and the argument can be repeated until the proof is complete.

THEOREM 3: *Some transform of* A *obeys the conditions of theorem* 2, *if and only if*

(i) *the sets* $\mathrm{R_k}$ *and* $\mathrm{C_k}$ *are well-defined for* $0 \leq \mathrm{k} < \mathrm{n}$,

(ii) *each* $\mathrm{R_k}$ *and each* $\mathrm{C_k}$ *is a chord, so that a unique permutation* ρ_k *is defined by* $\mathrm{R_k} = \mathrm{S(A,}\rho_k\mathrm{)}$, *and*

(iii) $\rho_k \rho_0^{-1} = (\rho_1 \rho_0^{-1})^k$ *for* $0 \leq \mathrm{k} < \mathrm{n}$.

If all the π_k of theorem 1 exist, then clearly $\pi_k = \rho_k$. Thus the real distinction between theorems 1 and 3 lies in conditions (i) and (ii), and the illustration just after the statement of theorem 2 was a "typical" one.

The necessity of the three conditions will be proved first.[11] Suppose transform $A' = (a'_{ij})$ of A obeys (a) and (b) of theorem 2. It will be shown initially that this implies, for each i and j,

$$a'_{i,\,i+q} > a'_{i,\,i+q-p} > \cdots > a'_{i,\,i+q-(n-1)p}, \tag{8}$$

$$a'_{j+s,\,j} < a'_{j+s-r,\,j} < \cdots < a'_{j+s-(n-1)r,\,j}; \tag{9}$$

only (8) need be discussed in detail. By (a) it follows that only $a'_{i,\,i+p}$ can be a largest entry of the ith row of A'. If p is relatively prime to n, then (8) follows readily from (a). If on the other hand there is a least residue k, with $1 \leq k < n$, such that n divides kp, then a contradiction is obtained[12] by choosing j different from $i+q-\nu p$ for $0 \leq \nu < k$ and using (a) of theorem 2 to write

$$a'_{ij} < a'_{i,\,j+p} < \cdots < a'_{i,\,j+kp} = a'_{ij}.$$

Thus (8) is proved.

By (8) and (9), each row and column of A' (and thus of A) has all its entries distinct; condition (i) has been verified. By the nature of row and column permutations, A will satisfy condition (ii) if A' does. Now the kth smallest entry in the ith row of A' is $a'_{i,\,i+q-(n-1-k)p}$; if some R_k were *not* a chord we would have

$$i+q-(n-1-k)p \equiv t+q-(n-1-k)p \pmod{n}$$

for distinct residues i and t, which is impossible. Similarly each C_k is a chord. Thus only the necessity of (iii) remains to be justified.

There exist permutations σ and τ such that eq (4) holds for all permutations π. In particular, by lemma 2,

$$R_k = S(A,\rho_k) = S(A',\tau\rho_k\sigma^{-1}),$$

$$C_k = S(A,\rho_k) = S(A',\tau\rho_k\sigma^{-1}).$$

Displays (8) and (9) show that

$$R_k = S(A',\delta_q(\delta_{(n-k-1)p})^{-1}) = S(A',\delta_q(\delta_p)^{k+1}),$$

$$C_k = S(A',(\delta_s)^{-1}\delta_{kr}) = S(A',(\delta_s)^{-1}(\delta_r)^k),$$

so that we have

$$\tau\rho_k\sigma^{-1} = \delta_q(\delta_p)^{k+1}, \tau\rho_k\sigma^{-1} = (\delta_s)^{-1}(\delta_r)^k \quad \text{for } k = 0, 1, \ldots, n-1. \tag{10}$$

Setting $k=0$ in the first of these equations yields $\sigma = (\delta_{p+q})^{-1}\tau\rho_0$, so that (10) yields

$$\rho_k\rho_0^{-1} = (\tau^{-1}\delta_p\tau)^k$$

and condition (iii) is satisfied.

For the *sufficiency* proof, suppose the three conditions hold. As in the proof of theorem 1, we define a mapping τ^{-1} of $\{0,1, \ldots, n-1\}$ into itself by

$$\tau^{-1}(k) = \rho_k\rho_0^{-1}(0) = (\rho_1\rho_0^{-1})^k(0),$$

and observe that τ^{-1} is a permutation so that τ is well-defined. We have

$$\delta_k(i) = \tau(\rho_1\rho_0^{-1})^{k+i}(0) = \tau(\rho_k\rho_0^{-1})\tau^{-1}(i)$$

so that $\delta_k = \tau(\rho_k\rho_0^{-1})\tau^{-1}$; equivalently $\rho_k\rho_0^{-1} = \tau^{-1}\delta_k\tau$. On defining $\sigma = (\delta_1)^{-1}\tau\rho_0$, we find that the *first* of eqs (10) holds with $p=1$ and $q=0$. By setting $r=1$ and $s=n-1$ we also satisfy the second of eq (10), so that conditions (a) and (b) of theorem 2 hold for the transform of A under row permutation σ and column permutation τ. Since

$$p+q+r+s = n+1$$

is relatively prime to n, the proof is complete.

Two comments are in order. First, the last paragraph shows that some transform of A obeys theorem 2 for *some* (p,q,r,s) if and only if at least one transform does so with $p=r=1$, $q=0$, $s=n-1$. Second, there is no need to apply the criterion of theorem 3 to $(-A)$ and $\pm A^T$ as well as A; the situation is just like the one mentioned directly after the proof of theorem 1.

---

[9] *A line* of a matrix is either a row or a column.

[10] R_0 consists of the smallest entry of the top row of A, the smallest entry of the next row, etc. R_1 consists of the next-to-smallest entry of the top row, the next-to-smallest entry of the next row, etc.

[11] In the necessity proof, the fact that $p+q+r+s$ is relatively prime to n is not used. This fact is actually a *consequence* of conditions (a) and (b) of theorem 2, which also (see eq (10)) imply that $p=r$ and that $p+q+s$ is divisible by n.

[12] The assumption $n > 1$ is tacit throughout.

4. Related Results

The proof of theorem 2 will obviously remain valid if p,q,r, and s are permitted to vary with i and j in appropriate ways. This leads to the sufficient condition for complete mixture contained in the following theorem.

THEOREM 4: *Let* p, q, r *and* s *be permutations of* $\{0, 1, \ldots, n-1\}$ *and consider a game with* n-by-n *matrix* A *such that*

$$\text{(a) } a_{ij} < a_{i,p(j)} \qquad \textit{whenever } j \neq q(i),$$

$$\text{(b) } a_{ij} < a_{r(i),j} \qquad \textit{whenever } i \neq s(j).$$

If rspq *is a cyclic permutation then no optimal* X^* *has a zero component, while if* pqrs *is cyclic then no optimal* Y^* *has a zero component.*

The proof is so like that of theorem 2 that it can be omitted; q and s must be permutations (and not merely mappings of $\{0, 1 \ldots, n-1\}$ into itself) to permit the same to be true of *rspq* or *pqrs*.

One would expect the criterion of theorem 4 (aided by the four operations listed in the introduction) to be more powerful than that of theorem 2 (similarly aided). In particular, it would seem possible that some transform of A (and thus A itself) could be proved to have no zero component in any optimal X^* by means of this criterion, and some *other* transform of A (and thus A itself) proved to have no zero component in any optimal Y^*. All these hopes are dashed by the following result.

THEOREM 5: *Some transform of* A *obeys* (*a*) *and* (*b*) *of theorem 4 for some* p,q,r,s *if and only if some transform of* A *obeys the condition of theorem 2.*

The correctness in the "if" direction is trivial, since theorem 2 is the special case of theorem 4 in which the relevant permutations are diagonal. So we need only assume that some transform $A' = (a'_{ij})$ of A obeys (a) and (b) of theorem 4, and deduce that A obeys the three conditions of theorem 3.

Condition (a) implies that only $q(i)$ can be left fixed by p; since $q(0) \neq q(1)$, *no* symbol is left fixed by p. Therefore, since by (a) no element of the ith row of A' *except* $a'_{i,q(i)}$ can be the row's largest entry, we see that the sequence of subscripts

$$(i, i+q), \qquad (i, i+q-p), \ \ldots \ (i, i+q-(n-1)p)$$

in (8) now becomes [13]

$$(i, q(i)), \qquad (i, p^{-1}q(i)), \ \ldots \ (i, p^{-(n-1)}q(i)). \quad (11)$$

If σ and τ are the row and column permutations leading from A to A', then we find that all ρ_k are well-defined with

$$\tau \rho_k \sigma^{-1} = p^{k+1-n}q \qquad \text{for } k = 0, 1, \ldots, n-1$$

in analogy with the first part of (10). From this it follows that condition (iii) of theorem 3 is obeyed.

Similarly, using (b) and the appropriate generalization of (9), we find that the C_k's are well-defined chords, so that (i) and (ii) of theorem 3 holds and the proof is complete.

Bohnenblust, Karlin, and Shapley (*op. cit.*, footnote 4) give a *second* sufficient condition for complete mixture, that of "main diagonal separated and dominant." To state it precisely, let $\mu = \mu(A)$ denote the largest of the entries of A off the main diagonal; then the condition consists of

$$a_{ii} > \mu \qquad \text{for } i = 0,1, \ \ldots \ ,n-1 \qquad (12)$$

and *either* of the situations

$$\sum_{i=0}^{n-1} a_{ij} > n\mu \qquad \text{for } j = 0,1,\ldots,n-1,$$

$$\sum_{j=0}^{n-1} a_{ij} > n\mu \qquad \text{for } i = 0,1,\ldots,n-1.$$

It is interesting to note that the matrix A_1 of example 1, which defied the BKS condition extended by theorem 1 but which yielded to our theorem 2, would also prove intractible to *this* condition. For S_0 and S_2 are not chords of A_1 (see the discussion of example 1), whereas it is easily shown that S_0 or S_{n-1} must be a chord if some transform of any of n-by-n $\pm A$, $\pm A^T$ is to obey (12).

It is natural to ask whether theorems 2 and 3 are sufficiently stronger than the BKS theorem to achieve identification of all completely mixed 3-by-3 games. This is unfortunately not the case; the completely mixed game whose payoff matrix is the A_2 of example 2, cited as a "maverick" by Bohnenblust, Karlin, and Shapley (*op. cit.*), is not tamed by our results. To see this, note that A_2 (and therefore $-A_2$ and $\pm A_2^T$ as well) fails to satisfy condition (i) of theorem 3.

A slight twist in the proof of theorem 2 leads to sharper analogs of theorems 2 and 4; for brevity, only the latter will be given (the former is obtained by specializing the relevant permutations to be diagonal).

THEOREM 6: *Let* p, q, r *and* s *be permutations of* $\{0,1, \ldots ,n-1\}$ *and consider a game with* n-*by*-n *matrix* A *such that*

$$\text{(a)} \quad a_{ij} \leq a_{i,p(j)} \qquad \textit{whenever } j \neq q(i),$$

$$\text{(a')} \quad a_{rsp(j),j} < a_{rsp(j),p(j)} \qquad \textit{for all } j,$$

$$\text{(b)} \quad a_{ij} > a_{r(i),j} \qquad \textit{whenever } i \neq s(j).$$

If rspq *is a cyclic permutation, then no optimal* X^* *has a zero component.*

THEOREM 7: *Let* p, q, r *and* s *be permutations of* $\{0,1, \ldots ,n-1\}$ *and consider a game with* n-*by*-n *matrix* A *such that*

[13] The detailed argument used earlier to justify (8) is easily generalized to support (11).

(a) $a_{ij} < a_{i,p(j)}$ *whenever* $j \neq q(i)$,

(b) $a_{ij} \geq a_{r(i),j}$ *whenever* $i \neq s(j)$,

(b′) $a_{i,pqr(i)} > a_{r(i),pqr(i)}$ *for all* i.

If pqrs *is a cyclic permutation, then no optimal* Y^* *has a zero component.*

Only theorem 6 need be proved. In view of the hypothesis on *rspq*, it suffices to prove that if $x_k^* = 0$ for some optimal X^*, then $x^*_{rspq(k)} = 0$ as well. By (1) and the optimality of X^*, we have

$$v \leq \sum_{i=0}^{n-1} x_i^* a_{i,q(k)} = \sum_{i \neq k} x_i^* a_{i,q(k)}.$$

In the right hand expression, all x_i^* are nonnegative and we have $a_{i,q(k)} \leq a_{i,pq(k)}$ for each term, with strict inequality for $i = rspq(k)$ (which is present since *rspq* does not leave k fixed). Thus, *unless* $x^*_{rspq(k)} = 0$ (i.e., if $x^*_{rspq(k)} > 0$), we have

$$v < \sum_{i \neq k} x_i^* a_{i,pq(k)} = \sum_{i=0}^{n-1} x_i^* a_{i,pq(k)},$$

and the rest of the proof is as for theorem 2, leading to the conclusion $x^*_{rspq(k)} = 0$.

Condition (b) of theorem 6 implies that each column of A has distinct entries, while condition (a) of theorem 7 implies that each row has distinct entries. Thus the matrix A_2 eludes these theorems (even when aided by the four operations). It remains unclear whether all 3-by-3 completely mixed games can be characterized along the lines pursued above.

Addendum. I am indebted to K. Goldberg (NBS Numerical Analysis Section) for the following observations.

If A is nonsingular, $v \neq 0$, and all row and column sums of A^{-1} are nonzero and have the same sign, then the game is completely mixed.[14] This covers examples A_1 and A_2 of section 2; it is not an "inspection" method since a matrix inversion is required. For $n = 3$, however, an "inspective" formulation can be given; if

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{pmatrix}$$

then what is required is that all 2-by-2 principal minors of

$$\begin{pmatrix} b_1 - c_1 & c_1 - a_1 & a_1 - b_1 \\ a_2 - b_2 & b_2 - c_2 & c_2 - a_2 \\ c_3 - a_3 & a_3 - b_3 & b_3 - c_3 \end{pmatrix} \text{ and } \begin{pmatrix} c_2 - b_3 & b_3 - a_1 & a_1 - c_2 \\ a_2 - c_3 & c_3 - b_1 & b_1 - a_2 \\ b_2 - a_3 & a_3 - c_1 & c_1 - b_2 \end{pmatrix}$$

have the same (strict) sign.

---

[14] K. Goldberg, Random notes on matrices, J. Research NBS **60** (1958) RP2850, Theorem 2 of that paper also can apply to the individual players.

(Paper 67B1–90)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

A New Type of Computable Inductor

Chester H. Page

(November 21, 1962)

The mutual inductance analog of the generalized Thompson-Lampard theorem (for cross capacitances) is developed. An infinitely long cage of five parallel wires can yield an absolute inductance of

$$10^{-7} \ln \frac{3+\sqrt{5}}{2}$$

henries per meter. End-effects of order $1/l^2$ occur in a finite cage, but can be reduced to order $1/l^4$ by using eight wires.

The eight-wire cage has the advantage of overdetermined relations among the inductances to be measured, allowing an estimate of experimental error in the calibration of a standard. Errors due to faulty cage geometry are shown to be of the order of 1 in 10^7.

The Thompson-Lampard theorem relates to the cross-capacitances of a cylindrical configuration of conductors. Consider a conducting cylindrical shell whose right cross section is an arbitrary closed curve, divided into four segments by infinitesimal gaps. The generalized theorem [1] states that the two cross capacitances, per unit length of the cylinder, are related by

$$e^{-4\pi^2 C_1/\Gamma_e} + e^{-4\pi^2 C_2/\Gamma_e} = 1$$

where Γ_e (often written ϵ_0) is the electric constant of the system of units considered.

The generality of this theorem suggests that an analogous theorem should exist for the mutual inductances of a cage of parallel wires. Consider first an infinitely long cage of four parallel wires, penetrating a transverse plane as in figure 1.

FIGURE 1

Three mutual inductances are involved; the pairs of loops whose traces in the plane are (1) $\overline{ab}$ and $\overline{cd}$, (2) $\overline{ad}$ and $\overline{bc}$, (3) $\overline{ac}$ and $\overline{bd}$. In case (1), the mutual inductance per unit cage length is

$$m = \frac{\Gamma_m}{2\pi} \ln \frac{\overline{ac}\ \overline{bd}}{\overline{ad}\ \overline{bc}} \tag{1}$$

where Γ_m is the magnetic constant, often written μ_0. By considering a, b, c, d to be points in a complex variable plane, we can write

$$m = \frac{\Gamma_m}{2\pi} \ln \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right|. \tag{2}$$

[1] D. G. Lampard and R. D. Cutkosky, Some results on the cross capacitances per unit length of cylindrical three-terminal capacitors with thin dielectric films on their electrodes, I.E.E. Monograph 351 M, 1960.

We shall find it convenient to use

$$m=-\frac{\Gamma_m}{2\pi}\ln|x|=-\frac{\Gamma_m}{4\pi}\ln A \tag{3}$$

where A is the square of the absolute value of the cross ratio x.

The desired theorem is a *nonhomogeneous* identity among the several mutual inductances. It can be expressed as an identity among the cross ratios. Thus the problem is to find a geometrical identity among the cross ratios of a set of points in a plane.

Now

$$x_1\equiv\frac{(a-d)(b-c)}{(a-c)(b-d)} \qquad x_2\equiv\frac{(a-b)(d-c)}{(a-c)(d-b)}$$

yielding the identity

$$x_1+x_2\equiv 1. \tag{4}$$

If the four points lie on a circle (including a straight line), the cross ratios are *real* (and positive), so that $|x_1|+|x_2|=1$, yielding

$$e^{-2\pi m_1/\Gamma_m}+e^{-2\pi m_2/\Gamma_m}=1 \tag{5}$$

corresponding to the Thompson-Lampard theorem. Unfortunately, departures from a circular locus make the x's complex, and eq (4) no longer implies any relation between the $|x|$'s.

The third possible pairing corresponds to crossed loops, with

$$x_3=\frac{(a-b)(c-d)}{(a-d)(c-b)}$$

and is related to the others through

$$x_3\equiv -x_2/x_1$$

yielding the *homogeneous* identity,

$$m_3=|m_2-m_1|$$

which is a circuit relation, independent of the cage configuration.

1. Pentagonal Cage

For a five-wire cage, there are five pairs of sides (fig. 2).

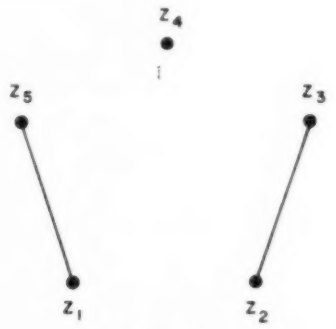

FIGURE 2
(Illustrating loops for M_1).

There are five mutual inductances corresponding to pairs of nonadjacent sides:

$$x_i \equiv \frac{(z_{i+1}-z_i)(z_{i+2}-z_{i-1})}{(z_{i+1}-z_{i-1})(z_{i+2}-z_i)}, \text{ indices modulo 5}$$

There are also five crossed-pairs (fig. 3):

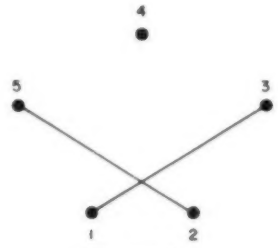

FIGURE 3
(Illustrating loops for N).

$$y_i \equiv \frac{(z_{i+2}-z_{i+1})(z_i-z_{i-1})}{(z_{i+2}-z_{i-1})(z_i-z_{i+1})}$$

related to the x_i through the identity

$$x_i y_i \equiv -x_{i-1}x_{i+1}$$

which yields the circuit relation

$$m_i+n_i=m_{i-1}+m_{i+1}.$$

The five x_i are mutually related by the nonhomogeneous identities

$$x_i+x_{i-1}x_{i+1}\equiv 1. \tag{6}$$

These identities will be shown to yield an identity among $|x_i|$, and therefore among the m_i. The result is the desired analog of the Thompson-Lampard theorem.

2. Derivation of Theorem for Pentagonal Cage

The identities (6) with $i=1$ and $i=2$ can be combined to yield

$$x_3+x_5-\frac{1}{x_1}-\frac{1}{x_2}=-\frac{x_1}{x_2}-\frac{x_2}{x_1}. \tag{7}$$

The identities (6) for $i=1,2,3,5$ yield directly

$$\begin{aligned}
|1-x_3|^2&=|x_2x_4|^2=A_2A_4\\
|1-x_5|^2&=|x_1x_4|^2=A_1A_4\\
\left|1-\frac{1}{x_1}\right|^2&=\left|\frac{x_2x_5}{x_1}\right|^2=A_2A_5/A_1\\
\left|1-\frac{1}{x_2}\right|^2&=\left|\frac{x_1x_3}{x_2}\right|^2=A_1A_3/A_2
\end{aligned} \tag{8}$$

where

$$A_i \equiv |x_i|^2.$$

667805—63——3

We also have the purely algebraic identity

$$|1-x_3|^2+|1-x_5|^2-\left|1-\frac{1}{x_1}\right|^2-\left|1-\frac{1}{x_2}\right|^2-|x_3|^2-|x_5|^2+\left|\frac{1}{x_1}\right|^2+\left|\frac{1}{x^2}\right|^2\equiv-2\,\mathrm{Re}\left(x_3+x_5-\frac{1}{x_1}-\frac{1}{x_2}\right). \quad (9)$$

Using (7) and (8), this becomes

$$A_2A_4+A_1A_4-A_2A_5/A_1-A_1A_3/A_2-A_3-A_5+1/A_1+1/A_2=2\,\mathrm{Re}\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)$$

$$=2\,\mathrm{Re}\left(\frac{x_1\overline{x_2}}{A_2}+\frac{x_2\overline{x_1}}{A_1}\right)$$

$$=2(\alpha_1\alpha_2+\beta_1\beta_2)\left(\frac{1}{A_1}+\frac{1}{A_2}\right) \quad (10)$$

where

$$x_i\equiv\alpha_i+j\beta_i, \qquad \overline{x_i}\equiv\alpha_i-j\beta_i.$$

Now identity (6) for $i=1$, $\quad x_1+x_2x_5\equiv1$

yields

$$2\alpha_1=1+A_1-A_2A_5$$

$$\beta_1^2\equiv|x_1|^2-\alpha_1^2=A_1-\alpha_1^2. \quad (11)$$

Solving (10) for $\beta_1\beta_2$, squaring for $\beta_1^2\beta_2^2$, and substituting from (11) for β_1^2, α_1, and from the similar relations for β_2^2, α_2, leads to lengthy algebraic manipulation which yields

$$\begin{aligned}A_1A_2A_3A_4A_5-A_1^2A_3A_4-A_2^2A_4A_5-A_3^2A_5A_1-A_4^2A_1A_2-A_5^2A_2A_3\\+A_1A_2A_4+A_2A_3A_5+A_3A_4A_1+A_4A_5A_2+A_5A_1A_3\\+A_1A_3+A_2A_4+A_3A_5+A_4A_1+A_5A_2-A_1-A_2-A_3-A_4-A_5+1\equiv0.\end{aligned} \quad (12)$$

Replacing each A_i by $e^{-4\pi m_i/\Gamma_m}$ yields the mutual inductance identity.

If the pentagon is regular, the A_i are equal, and (12) can be factored:

$$(A^2-3A+1)^2(A+1)=0. \quad (13)$$

The allowable roots are

$$A=\frac{3\pm\sqrt{5}}{2}$$

corresponding to

$$m=\frac{\Gamma_m}{4\pi}\ln\frac{3+\sqrt{5}}{2}.$$

(The two roots are reciprocal, so yield the same absolute value of mutual inductance.)

3. Application

In principle, we measure the approximately equal m_i by comparison with a standard, m. The assumed value of the standard is in error by an unknown factor $(1+\delta)$. The object of the experiment is to determine δ.

If

$$m_i=a_im=\frac{a_i}{1+\delta}(1+\delta)m$$

the apparent, or measured values, are

$$\hat{m}_i=\frac{a_im}{1+\delta}=\frac{m_i}{1+\delta}. \quad (14)$$

Substitution into the theorem (12) yields an implicit equation for δ in terms of the observed $\hat{m}_i$.

In practice, there will be random measurement errors associated with each $\hat{m}_i$. These will be treated in a later section. A more serious problem is presented by the fact that we cannot measure the inductance per unit length of an infinite cage, but only the change with the length of a finite cage—i.e., end effects must be considered.

4. End Effects

Consider two loops of length l, the loop planes cutting the transverse plane (fig. 4) in the lines ρ_{i-1} and ρ_{i+1}.

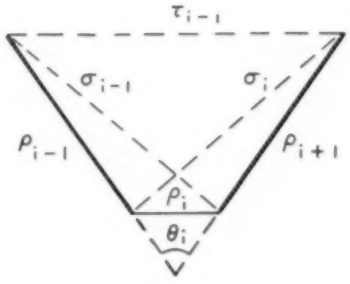

FIGURE 4

The mutual inductance is given by the double line integral around the two loops:

$$M_i=\frac{\Gamma_m}{4\pi}\oint\oint\frac{d\boldsymbol{\lambda}_1\cdot d\boldsymbol{\lambda}_2}{R_{12}}$$

End effects arise from the truncation of the cage at length l, and also from the couplings among the shorting bars of length ρ_{i-1} and ρ_{i+1} at the cage ends. The expression for M_i becomes:

$$\frac{4\pi M_i}{\Gamma_m}=\int_0^l\int_0^l dxdy\{[(x-y)^2+\rho_i^2]^{-1/2}+[(x-y)^2+\tau_{i-1}^2]^{-1/2}-[(x-y)^2+\sigma_{i-1}^2]^{-1/2}$$
$$-[(x-y)^2+\sigma_i^2]^{-1/2}\}+2\cos\theta_i\int_0^{\rho_{i-1}}\int_0^{\rho_{i+1}}\left\{\frac{1}{r}-\frac{1}{\sqrt{l^2+r^2}}\right\}dudv \quad (15)$$

where r is the distance in the transverse plane between point u on ρ_{i-1} and point v on ρ_{i+1}.

Since we are considering inductance per unit length, we are interested in

$$m_i\equiv\frac{\Delta M_i}{\Delta l}=\frac{\partial M_i}{\partial l}+\frac{\Delta l}{2}\frac{\partial^2 M_i}{\partial l^2}$$

particularly when l is large compared to the cross dimensions of the cage. We have

$$\frac{4\pi}{\Gamma_m}\frac{\partial M_i}{\partial l}=2\int_0^l dx\{[(l-x)^2+\rho_i^2]^{-1/2}+[(l-x)^2+\tau_{i-1}^2]^{-1/2}-[(l-x)^2+\sigma_{i-1}^2]^{-1/2}$$
$$-[(l-x)^2+\sigma_i^2]^{-1/2}\}+2l\cos\theta_i\int_0^{\rho_{i-1}}\int_0^{\rho_{i+1}}dudv(l^2+r^2)^{-3/2}=\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}$$
$$+2\ln\frac{(l+\sqrt{l^2+\rho_i^2})\,(l+\sqrt{l^2+\tau_{i-1}^2})}{(l+\sqrt{l^2+\sigma_i^2})\,(l+\sqrt{l^2+\sigma_{i-1}^2})}+2l\cos\theta_i\int_0^{\rho_{i-1}}\int_0^{\rho_{i+1}}dudv(l^2+r^2)^{-3/2}. \quad (16)$$

For large l, this is approximated by

$$\frac{4\pi}{\Gamma_m}\frac{\partial M_i}{\partial l}\doteq\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}+\frac{\rho_i^2+\tau_{i-1}^2-\sigma_{i-1}^2-\sigma_i^2}{2l^2}+\frac{2\rho_{i-1}\rho_{i+1}\cos\theta_i}{l^2}-\frac{3}{16l^4}(\rho_i^4+\tau_{i-1}^4-\sigma_{i-1}^4-\sigma_i^4)$$
$$-\frac{3}{2}\frac{\rho_{i-1}\rho_{i+1}\cos\theta_i}{l^4}\left\{\tau_{i-1}^2+\rho_i^2+\rho_{i-1}\rho_{i+1}\cos\theta_i+\frac{3}{2}(\rho_{i-1}^2+\rho_{i+1}^2)\right\}. \tag{17}$$

The distances in figure 4 are related by

$$\rho_i^2+\tau_{i-1}^2-\sigma_{i-1}^2-\sigma_i^2=-2\rho_{i-1}\rho_{i+1}\cos\theta_i$$

simplifying (17) to

$$\frac{4\pi}{\Gamma_m}\frac{\partial M_i}{\partial l}\doteq\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}+\frac{\rho_{i-1}\rho_{i+1}\cos\theta_i}{l^2}+\frac{1}{l^4}\left\{\frac{3}{16}(\rho_i^4+\tau_{i-1}^4-\sigma_{i-1}^4\right.$$
$$\left.-\sigma_i^4+4\rho_i^2\tau_{i-1}^2-4\sigma_{i-1}^2\sigma_i^2)-(\rho_{i-1}^2+\rho_{i+1}^2)\rho_{i-1}\rho_{i+1}\cos\theta_i\right\}. \tag{18}$$

The second derivative, $\partial^2 Mi/\partial l^2$, is easily found and leads to

$$\frac{4\pi}{\Gamma_m}\frac{\Delta M_i}{\Delta l}\doteq\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}+\frac{\rho_{i-1}\rho_{i+1}\cos\theta_i}{l^2}\left(1-\frac{2\Delta l}{l}\right)+\frac{a}{l^4}\left(1-\frac{4\Delta l}{l}\right) \tag{19}$$

where a stands for the coefficient of $1/l^4$ in eq (18).

For large Δl, integration of (18) yields

$$\frac{4\pi}{\Gamma_m}\frac{M_i(l_2)-M_i(l_1)}{l_2-l_1}\doteq\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}+\frac{\rho_{i-1}\rho_{i+1}\cos\theta_i}{l_1l_2}+\frac{a}{3}\frac{l_1^2+l_1l_2+l_2^2}{l_1^3l_2^3}. \tag{19a}$$

For a regular pentagon, $\sigma^2=\tau^2=\dfrac{3+\sqrt{5}}{2}\delta^2$, $\qquad\theta=\pi/5$,

yielding

$$\frac{4\pi}{\Gamma_m}\frac{\partial M}{\partial l}\doteq\ln\frac{3+\sqrt{5}}{2}+\frac{1+\sqrt{5}}{4}\rho^2/l^2. \tag{20}$$

The ρ^2/l^2 end-effect is intolerable. Equation (19) suggests the use of an octagon, $\cos\theta=0$. For a regular octagon,

$$\sigma^2=\rho^2(2+\sqrt{2}),\qquad\tau^2=\rho^2(3+2\sqrt{2}) \tag{21}$$

yielding

$$\frac{4\pi}{\Gamma_m}\frac{\partial M}{\partial l}\doteq\ln 2-3\frac{(3+2\sqrt{2})}{8}\rho^4/l^4 \tag{22}$$

with a tolerable end-effect. Errors due to irregularity of the octagon will be treated later.

5. Octagonal Cage

Since cross ratios are invariant under bilinear transformations, three points of a polygonal cage can be chosen arbitrarily. This implies that there are $2(n-3)$ independent mutual inductances. For the pentagon, $2(n-3)=4$, providing one identity among the five mutual inductances. In the case of an octagon, there are 10 independent mutual inductances, so the 8 pairs of mutually perpendicular sides are insufficient. There are, however, additional pairs of perpendicular loops (fig. 5). Fortunately, the dotted pair and the solid pair have the same nominal value of mutual inductance. Among the 16 mutual inductances now under consideration, there are two circuit identities, leaving 14 essentially independent measurements. This is a sufficient overdetermination of the cross ratios of the cage to yield a reasonable estimate of the validity of the final determination of $\hat{\delta}$ in the presence of random errors of measurement.

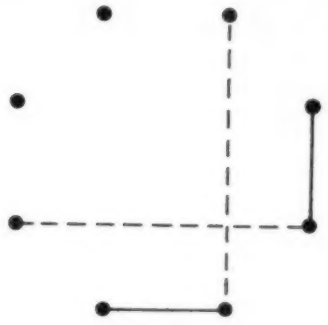

FIGURE 5

For the two sets of mutual inductances per unit length, of an infinite cage, we have

$$m_i=\frac{\Gamma_m}{4\pi}\ln\frac{\sigma_{i-1}^2\sigma_i^2}{\tau_{i-1}^2\rho_i^2}$$

$$n_i=\frac{\Gamma_m}{4\pi}\ln\frac{\sigma_{i-2}^2\sigma_{i+1}^2}{\rho_i^2\tau_{i+3}^2},\ \text{indices mod 8.} \tag{23}$$

It does not seem feasible to express the geometric identity among the cross ratios explicitly, as was done for the pentagon eq (12). There is, however, a useful procedure for the *application* of the inherent identity, i.e., for the determination of the calibration constant, δ. We attribute all random measurements errors to random variations of the standard, i.e., to variations in δ. This artifice simplifies the problem and at the same time yields a conservative estimate of the reliability of the calibration of the standard inductor.

We modify eq (14) to the form

$$\hat{m}_i=m_i/(1+\delta_i)$$

$$\hat{n}_i=n_i/(1+\delta_{i+8}),\qquad i=0\ldots 7 \tag{24}$$

to allow for errors in the 16 inductance determinations. If there were no experimental errors, all δ_i would be equal, and would be overdetermined by the 16 eq (24). A computer program could be developed to assign positions to the cage wires that would make eq (24) compatible, thus yielding both m_i and δ. In the presence of errors, our best estimate of δ arises from assigning cage coordinates to minimize the variance of the resulting δ_i.[2]

$$\varphi=\frac{1}{16}\sum_0^{15}(\delta_i-\delta)^2$$

$$\varphi_{\min}=s^2=\overline{\delta_i^2}-\bar{\delta}_i^{\,2},\qquad \delta=\overline{\delta_i}. \tag{25}$$

Cage coordinates are to be assigned to minimize s^2. From eq (24), we have

$$s^2=\frac{1}{16}\sum_0^7\left(\frac{m_i-\hat{m}_i}{\hat{m}_i}\right)^2+\frac{1}{16}\sum_0^7\left(\frac{n_i-\hat{n}_i}{\hat{n}_i}\right)^2-\left\{\frac{1}{16}\sum_0^7\frac{m_i-\hat{m}_i}{\hat{m}_i}+\frac{1}{16}\sum_0^7\frac{n_i-\hat{n}_i}{\hat{n}_i}\right\}^2. \tag{26}$$

[2] This procedure was suggested by Dr. R. J. Arms.

The theoretical values m_i, n_i, are computed from assigned cage coordinates using eq (23). Since the m_i and n_i depend only upon cross ratios, and cross ratios are unaffected by bilinear transformations, three of the cage points can be arbitrarily assigned. The other five are then adjusted to minimize s^2. When s^2 is minimized, the term in braces in (26) is the desired value of δ. The measured values $\hat{m}_i$ and $\hat{n}_i$ will be essentially equal, so this term can be approximated by

$$\delta \doteq \frac{\Sigma m_i + \Sigma n_i}{\Sigma \hat{m}_i + \Sigma \hat{n}_i} - 1. \tag{27}$$

For displacement of cage wires from the vertices of a regular octagon, both Σm_i and Σn_i are at a saddle point. In fact, for displacement of any one wire in a direction making an angle φ with the radius from the octagon center, we have

$$\frac{\partial \sum_i m_i}{\partial (r/\rho)} = \frac{\partial \sum_i n_i}{\partial (r/\rho)} = 0 \tag{28a}$$

$$\frac{\partial^2 \sum_i m_i}{\partial (r/\rho)^2} = \frac{\partial^2 \sum_i n_i}{\partial^2 (r/\rho)^2} = -\frac{\Gamma_m}{\pi} (2-\sqrt{2}) \cos 2\varphi. \tag{28b}$$

Thus an ignored displacement of one wire from its regular position would affect s^2, but not δ, to first order. The second order effect on δ vanishes if averaged over all directions of displacement.

6. End Effects

The use of a regular octagon reduces end effects to terms in $1/l^4$. Departures from regularity introduce end effects in $1/l^2$. Formulas (18) and (19) in M_t are converted to the corresponding formulas in N_t by the following changes:

$$\sigma_{t-1} \to \sigma_{t-2} \qquad \rho_t \to \rho_t \qquad \theta_t \to \varphi_t$$
$$\sigma_t \to \sigma_{t+1} \qquad \rho_{t-1} \to \tau_{t-2}$$
$$\tau_{t-1} \to \tau_{t+3} \qquad \rho_{t+1} \to \tau_t$$

where φ_t is the angle between τ_{t-2} and τ_t (fig. 6).

The computed m_i and n_i used for minimizing s^2 eq (26) can be taken from eq (19) and its n_i equivalent, thus automatically allowing for end effects. Ignoring these end effects by using the infinite-cage formula in the minimization would still produce only a small error.

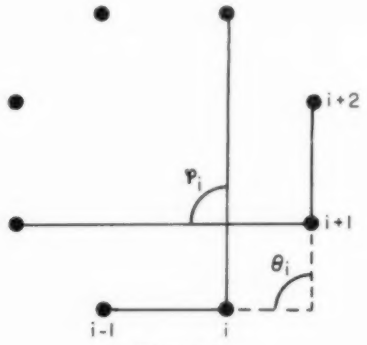

FIGURE 6

For a regular octagon, eq (21) yields a fractional error of

$$-\frac{3(3+2\sqrt{2})}{8 \ln 2} \rho^4/l^4$$

in m_i, and the corresponding formula for n_i yields a fractional error of

$$-\frac{3(3+2\sqrt{2})^3}{8 \ln 2} \rho^4/l^4.$$

The resulting fractional error in $\Sigma(m_i+n_i)$ is approximately $-7\rho^4/l^4$, which is of the order of 10^{-7} for $\rho/l=0.01$.

We must also investigate the error arising from the $1/l^2$ term for a slightly irregular octagon. For the m_i, we have

$$\epsilon_i=\rho_{i-1}\rho_{i+1} \cos \theta_i/l^2 \ln 2$$

and for the n_i, $\epsilon_i=\tau_{i-2}\tau_i \cos \phi_i/l^2 \ln 2$. By eq (27), these terms lead to an erroneous δ, or fractional error in the mutual inductance assigned to the standard, equal to $\bar{\epsilon}_i$:

$$\delta_e=\frac{\Sigma\epsilon_i}{16}=\frac{\Sigma(\boldsymbol{\rho}_{i-1}\cdot\boldsymbol{\rho}_{i+1}+\boldsymbol{\tau}_{i-2}\cdot\boldsymbol{\tau}_i)}{16l^2 \ln 2}.$$

If points 1, 2, 3 are fixed, and the other five displaced from nominal by $\boldsymbol{\eta}_i$, we find

$$\Sigma\epsilon_i=\frac{2}{l^2 \ln 2}[2\boldsymbol{\eta}_5\cdot\boldsymbol{\eta}_7+2\boldsymbol{\eta}_6\cdot\boldsymbol{\eta}_8+\boldsymbol{\eta}_4\cdot\boldsymbol{\eta}_6-\boldsymbol{\eta}_4\cdot\boldsymbol{\eta}_5-\boldsymbol{\eta}_4\cdot\boldsymbol{\eta}_7-\boldsymbol{\eta}_5\cdot\boldsymbol{\eta}_6-\boldsymbol{\eta}_5\cdot\boldsymbol{\eta}_8-\boldsymbol{\eta}_6\cdot\boldsymbol{\eta}_7-\boldsymbol{\eta}_7\cdot\boldsymbol{\eta}_8].$$

If

$$|\boldsymbol{\eta}| \leq \alpha, \text{ then } |\Sigma\epsilon_i| \leq \frac{2}{l^2 \ln 2} 11\alpha^2$$

and

$$|\delta_e| \leq 2\alpha^2/l^2.$$

For

$$\alpha \sim 0.02\rho \text{ and } \rho \sim 0.01l, \ |\delta_e| \leq 10^{-7}.$$

7. Other Errors

A bent wire, with zero average displacement, is expected to yield an error similar to that of an ignored displacement, but of lower order. An ignored displacement affects $s^2(\delta)$ but not $\bar{\delta}$. This would have more effect on the reliability of calibration than on the calibration constant itself. Exact analysis of the bent wire situation is difficult; the effect should be checked experimentally by adding known perturbations to the cage.

The analysis has assumed ideal wires. Actual construction requires finite diameters. It is well known (and easily shown) that at zero frequency, such wires are equivalent to ideal wires. Proximity effects on the current distribution can be experimentally lumped with skin effect, and the net effect evaluated by making measurements at various frequencies. Nongeometrical effects, such as produced by the permeability of the wire and supports, will be investigated in the future.

8. Conclusion

It has been shown that there is a mutual inductance analog of the Thompson-Lampard theorem. In particular, a design is proposed wherein end effects are substantially reduced. The mutual inductance of this design is ln 2/10 μh/m.

(Paper 67B1–91)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

Numerical Computation of the Temporal Development of Currents in a Gas Discharge Tube*

W. Börsch-Supan** and H. Oser

(November 16, 1962)

The behavior of electrical currents in a gas discharge tube including space charge effects is investigated by numerical integration of the governing nonlinear partial differential equations. Both stationary solutions and the temporal development, under the influence of space charge effects, are considered. It is found that the truncation error can be greatly reduced by comparison with formal solutions for constant fields. The discussion is essentially restricted to the more mathematical questions.

1. Introduction

The behavior of electron and ion currents in a gas discharge tube as a function of time and the applied voltage has been investigated by several authors [1, 2, 3, 4][1]. Most of these have omitted the effect of space charge, but have estimated when the effect appears. Space charge, however, results in a temporally growing distortion of the electrical field and, therefore, in a severe nonlinearity of the equations governing the behavior of electron and ion currents in the tube. A. L. Ward [5] suggested the numerical integration of the nonlinear equations on an electronic computer for an essentially "one-dimensional" tube, i.e., a tube whose electrodes are parallel plates of dimensions large compared to their separation.

This paper describes the mathematical treatment of the basic equations necessary for the applicability of numerical methods. For the sake of completeness, a short derivation of the basic equations (sec. 2) and the formal treatment of the case of constant field (sec. 5) is given, although many of these considerations can be found in other papers too. The stationary case is treated extensively in section 3. The difference schemes used for the time-dependent case are discussed in section 4. The discussion of the results in section 6 is restricted to the more mathematical questions like the influence of truncation errors, and certain other errors occurring during the computations. A discussion of the physical significance of the results is given by A. L. Ward [10, 11].

2. Basic Equations

We state the equations in an Eulerian coordinate system, denoting the space coordinate by x and the time by t. The cathode is located at $x=0$, the anode at $x=d$. Let n_+, n_-, j_+, j_-, v_+, v_- be the density of positive ions (number of particles per unit of volume), the density of electrons, the current density of ions (electric charge passing through a cross section of unit area per unit of time), the current density of electrons, the drift velocities of ions and electrons, respectively. The ion current density and the ion velocity are counted positive if the ions are moving toward smaller x, the electron current density and the electron velocity are counted positive if the electrons are moving toward larger x. We denote the intensity of the electrical field by E, counting it positive if directed from the anode to the cathode. Let $\alpha=\alpha(E)$ be

*This work was sponsored by the Harry Diamond Laboratories, U.S. Department of the Army.
**Present address: Institut für Praktische Mathematik, Technische Hochschule Darmstadt, Germany.
[1] Figures in brackets indicate the literature references at the end of this paper.

the number of ionizations caused by each electron per unit length of its path, $q_+=\epsilon n_+$ the charge density of positive ions, $q_-=en_-$ the charge density of electrons, counted positive all the time, e being the elementary charge. With ϵ_0 denoting the dielectric constant, the processes in the tube can be described by the following equations:

(a) Continuity (Townsend's equations)

$$\frac{\partial q_-}{\partial t}=\alpha j_- -\frac{\partial j_-}{\partial x} \tag{2.1}$$

$$\frac{\partial q_+}{\partial t}=\alpha j_- +\frac{\partial j_+}{\partial x}. \tag{2.2}$$

(b) The electrostatic Maxwell's equation (Poisson's equation)

$$\epsilon_0 \frac{\partial E}{\partial x}=q_- -q_+. \tag{2.3}$$

The outer circuit supplying the voltage for the tube can be described by a capacitance C parallel to the tube and a resistance R in series to both the tube and the capacitance. The outer voltage applied to this system as shown in figure 1 is called U. If V is the voltage across the gap of the tube and I the current to and from the tube, then the equation

$$U=R\left(I+C\frac{dV}{dt}\right)+V \tag{2.4}$$

describes the behavior of the outer circuit. The current I can be obtained from the mean current in the tube and the change in time of the voltage across the gap by the following considerations. Let us introduce the abbreviations $q=q_- -q_+$ and $j=j_-+j_+$. Furthermore let S denote the area of a cross section of the current in the tube, d the distance between cathode and anode, and Q the charge accumulated on a unit area of the cathode. Then the law of conservation of charge, applied to the cathode yields

$$I=S\left[j(0,t)-\frac{dQ}{dt}\right].$$

Now the charge Q is connected with the voltage across the gap by

$$Q=-\epsilon_0 E(0,t)$$

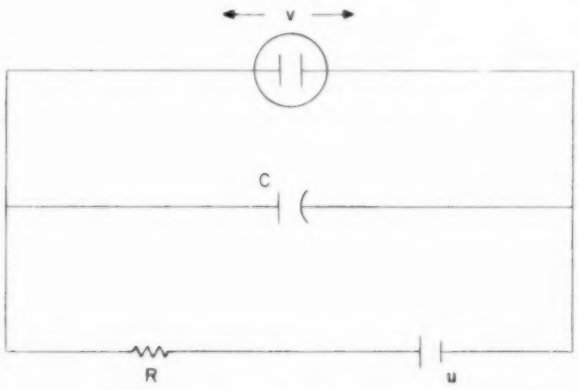

FIGURE 1. *Diagram of the electric circuit.*

and the equation

$$V=\int_0^d Edx=dE(0,t)+\epsilon_0^{-1}\int_0^d\int_0^x q(\xi,t)d\xi dx \tag{2.5}$$

following from eq (2.3) by integrating twice. If we differentiate eq (2.5) with respect to time and use $\frac{\partial q}{\partial t}=-\frac{\partial j}{\partial x}$, we obtain

$$\frac{dE(0,t)}{dt}=\frac{1}{d}\frac{dV}{dt}+\frac{1}{d\epsilon_0}\int_0^d j(x,t)dx-\frac{1}{\epsilon_0}j(0,t)\cdot \tag{2.6}$$

Now the current I can be expressed as

$$I=S\left(\frac{\epsilon_0}{d}\frac{dV}{dt}+\frac{1}{d}\int_0^d jdx\right)\cdot$$

Introducing this into eq (2.4) leads to the following ordinary differential equation for V:

$$R\left(C+\frac{S\epsilon_0}{d}\right)\frac{dV}{dt}=U-V-RSJ \tag{2.7}$$

where

$$J=\frac{1}{d}\int_0^d jdx$$

is the mean current density in the gap. We write: $J=qv$, where $v=v_+v_-/(v_-+v_+)$ is an average velocity. Instead of eliminating $\partial E(0,t)/\partial t$ one may eliminate dV/dt by using eq (2.6). Then, instead of eq (2.7), one obtains an ordinary differential equation in time for $E(0,t)$:

$$R(Cd+S\epsilon_0)\frac{dE(0,t)}{dt}=U-V+\frac{RCd}{\epsilon_0}J-R\left(\frac{Cd}{\epsilon_0}+S\right)j(0,t) \tag{2.8}$$

where V follows from eq (2.5).

The differential equation (2.7) or (2.8) furnishes the boundary condition corresponding to eq (2.3). For eqs (2.1) and (2.2) separate boundary conditions will have to be established. They describe the electron current at the cathode and the ion current at the anode respectively. The latter current is zero since there are no ions coming out of the anode:

$$j_+(d,t)=0. \tag{2.9}$$

The electron current at the cathode is given by

$$j_-(0,t)=j_p+\gamma_i j_+(0,t)+\gamma_p\int_0^d \sigma j_-dx \tag{2.10}$$

where j_p denotes the current density resulting from externally irradiating the cathode with photons. The second term describes electrons produced by ions hitting the cathode, γ_i being the probability that an incoming ion produces an outgoing electron. The last term comes from internally produced photons hitting the cathode. These "secondary" photons are assumed to be emitted from molecules which were excited by electron collisions. The number of excited molecules produced per unit of the path of a single electron is called $\sigma=\sigma(E)$, γ_p is the probability that a secondary photon produces an outgoing electron. It is assumed in eq (2.10) that the emission of photons occurs immediately after the excitation; however, the computer program contains a provision for an arbitrary time delay, simulating a delayed photoemission.

Equations (2.1), (2.2), (2.3), (2.5), (2.7), (2.9), and (2.10) together with initial distributions of q_+ and q_- and the initial voltage V determine the solution of the problem completely for times greater than the initial time up to infinity or to a certain time limit, provided the velocities v_+ and v_- and the ionization rates α and σ are given quantities.

Measurements show that v_+, v_-, and α can be approximated by the following types of functions of E and the pressure p in the tube:

$$v_+ = \begin{cases} \mu_+(1-B_1|E|/p)\dfrac{E}{p} & \text{if } |E|<W_1p \\ (B_3\sqrt{p/|E|}-B_2p^2/E^2)E/p & \text{if } |E|\geq W_1p \end{cases} \tag{2.11}$$

$$v_- = \mu_- E/p \tag{2.12}$$

$$\alpha = \begin{cases} C_1p\exp(-D_1p/|E|) & \text{if } |E|<W_2p \\ C_2p\exp(-D_2p/|E|) & \text{if } |E|\geq W_2p \end{cases} \Big\} \text{for molecular gases}$$
$$\alpha = \begin{cases} C_3p\exp(-D_3\sqrt{p/|E|}) & \text{if } |E|<W_2p \\ C_4p\exp(-D_4\sqrt{p/|E|}) & \text{if } |E|\geq W_2p \end{cases} \Big\} \text{for atomic gases} \tag{2.13}$$

where μ_+, μ_-, the B_i, C_i, D_i, and W_i are certain constants. The quantity σ is approximated by the same type of function as α with possibly different constants.

3. Steady State Solutions

The steady state has been investigated by several authors. We report here on calculations of space charge distributions in cold cathode discharge tubes, which have been conducted for a number of years at the National Bureau of Standards. A. L. Ward, who suggested this program, reported on the results in several publications [4], [10], [11], and we shall confine ourselves here to stating the equations and the method to solve them.

3.1. Differential Equations and Boundary Conditions

We assume a state of equilibrium, which means

$$\frac{\partial q_-}{\partial t}=0 \text{ and } \frac{\partial q_+}{\partial t}=0$$

at all times, and eq (2.1) reduces to

$$\frac{dj_-}{dx}=+\alpha(E)j_-(x) \tag{3.1}$$

which is Townsend's steady state equation. In order to take account of collisions between electrons and metastable molecules inside the tube, we add an extra term which is proportional to $j_-^2(x)$:

$$\frac{dj_-}{dx}=+\alpha(E)j_-(x)+\beta(E)j_-^2(x). \tag{3.2}$$

Here $\alpha(E)$ is defined by eq (2.13) and $\beta(E)$ is assumed to follow the same law, with possibly different parameters C_i, D_i, and W_2.

For eq (2.2), which governs the ions, we assume that the total current density is constant:

$$j_-(x)+j_+(x)=j=\text{const.} \tag{3.3}$$

Equation (2.3) is then written as follows:

$$\epsilon_0 \frac{dE}{dx} = \frac{p}{E} \cdot \begin{cases} \dfrac{j(x) - j_-(x)}{\mu_+(1 - B_1|E|/p)} & \text{if } |E| < W_1 p \\ \dfrac{j(x) - j_-(x)}{B_3\sqrt{\dfrac{p}{|E|}} - B_2\left(\dfrac{p}{E}\right)^2} & \text{if } |E| \geq W_1 p. \end{cases} \tag{3.4}$$

Equations (3.2) and (3.4) are two ordinary first-order differential equations for $E(x)$ and $j_-(x)$, provided that we are given the total current density function j.

Physically accessible are the currents at the electrodes, leading to the following boundary conditions

$$j_-(d) = j \text{ at the anode.} \tag{3.5}$$

$$j_-(0) = (j_p + \gamma j_+(d))/(1+\gamma) \text{ at the cathode.} \tag{3.6}$$

Here j_p is a contribution to the electron current density caused by external radiation and γ is a secondary ionization coefficient assumed to be a constant.

3.2. Integration of the System of Differential Equations

We distinguish here two cases:

(i) The current is so small that $\left|\frac{dE}{dx}\right| \ll \frac{E}{d}$ everywhere, i.e., we assume a constant field. $\alpha(E)$ and $\beta(E)$ are then constant also and eq (3.2) becomes a Bernoulli equation with constant coefficients, which can be integrated explicitly. The value of E is then determined from the second boundary condition.

(ii) Arbitrary large currents $j(x)$. The solution has to be found iteratively:

Starting at $x = d$ with $j_-(d) = j$, a value of $E(d)$ has to be assumed. (It seems most feasible to generate a whole family of solutions with increasing total current densities. The initial value $E(d)$ is then taken to be the solution of the previous case. For sufficiently small $j(x)$ assumption (i) holds and no difficulty arises in finding a starting value.)

Equations (3.2) and (3.4) are integrated simultaneously by means of a Runge-Kutta scheme. Iteration on $E(d)$ is performed until $j_-(0)$ is in sufficient agreement with the prescribed boundary condition.

As the total current densities increase, it becomes more and more difficult to find the proper $E(d)$. At $j_-(d) \approx 10^{-4}$ amp/cm^2 it was impossible to find a solution, even by the interval halving method.

Fortunately this limit covers most of the experimental data as far as the basic equations are valid. For the results we refer to the aforementioned publication of A. L. Ward.

4. Difference Equations

4.1. Difference Equations With Respect to Both Time and Space. Stability Considerations

In the most general case, it is not possible to give an exact solution of the system of equations described in section 2. Therefore, one tries to find an approximate numerical solution. The most convenient way to obtain a "numerical solution" is to introduce finite differences in time and space and to solve numerically the finite equations generated in this way for a certain set of values of the different parameters of the problem.

The way to transform the differential equations into difference equations is largely determined by requiring simplicity of the computational scheme and "stability" of the difference scheme. For the sake of simplicity we ask for explicit schemes as far as possible. For the same reason we use difference operators of the same order as the corresponding differential

operators. This ensures that the special computations at the boundaries are kept to a minimum. By requiring stability we exclude certain difference schemes which would lead to large amplification of any small errors (as rounding errors) when the time increases, at least in the limit of vanishing meshwidth. We use here the concept of stability as introduced by Lax and Richtmyer [8, 12].

Denoting by Δt and Δx the increments in time and space respectively, the difference equations corresponding to eqs (2.1) and (2.2) are

$$\frac{q_-(x, t+\Delta t)-q_-(x, t)}{\Delta t}=\alpha(x, t)j_-(x, t)-\frac{j_-(x, t)-j_-(x-\Delta x, t)}{\Delta x} \tag{4.1}$$

for $x=\Delta x, 2\Delta x, \ldots, M\Delta x=d$; $t=t_0, t_0+\Delta t, t_0+2\Delta t, \ldots$

$$\frac{q_+(x, t+\Delta t)-q_+(x, t)}{\Delta t}=\alpha(x, t)\cdot j_-(x, t)+\frac{j_+(x+\Delta x, t)-j_+(x, t)}{\Delta x} \tag{4.2}$$

for $x=0, \Delta x, 2\Delta x, \ldots, (M-1)\Delta x$; $t=t_0, t_0+\Delta t, t_0+2\Delta t, \ldots$.

The difference quotients with respect to x have been chosen unsymmetrically for the sake of stability, see [8, 12]. In order to obtain stability in the sense of Richtmyer for $\Delta t\to 0$, $\Delta x\to 0$, with $\frac{\Delta t}{\Delta x}=$const., we must require that

$$\Delta t<\frac{\Delta x}{\max\{v_-, v_+\}}, \tag{4.3}$$

and that both v_- and v_+ are nonnegative. Since v_- is much larger than v_+, this means that the timestep must not exceed the time an electron needs to go from one meshpoint in space to the next. This time is very short for many of the interesting developments in a gas discharge tube. Therefore, for many phenomena, it will be sufficient to assume that the electron density and current distribution will be in a quasi-equilibrium state, i.e., we replace the left hand side of eq (2.1) by zero. Since α is rather large in the interesting cases, we do not use the difference scheme obtained from eq (4.1) by putting the left hand side equal to zero. We rather integrate eq (2.1) formally:

$$j_-(x, t)=j_-(0, t)\cdot\exp\left\{\int_0^x \alpha(x', t)dx'\right\}$$

and replace the integral in the argument of the exponential function by a finite sum, using the trapezoidal rule:

$$j_-(m\cdot\Delta x, t)=j_-(0, t)\cdot\exp\left\{\sum_{k=0}^{m-1}\frac{1}{2}[\alpha(k\cdot\Delta x, t)+\alpha((k+1)\Delta x, t)]\Delta x\right\} \tag{4.4}$$

for $m=1, 2, \ldots M$.

The stability condition for the system eq (4.4), (4.2) is now

$$\Delta t\leq\frac{\Delta x}{v_+}, \tag{4.5}$$

and $v_+\geq 0$. Thus one can use much larger timesteps than for the original system, namely, the timestep must not exceed the time an ion needs to go from one meshpoint to the next.

Equation (2.3) does not contain time derivatives. It shows that E, and therefore also α, are obtained by integrating q. Hence, stability is not affected, and the question how to choose a proper difference scheme replacing this equation can be separated from the question how to choose the timestep.

Of course, the characteristic time of the outer circuit will provide another bound for the timestep. As long as the term containing J is unessential in eq (2.7), this characteristic time is apparently $[R(C+S\epsilon_0/d)]$. If one uses instead of eq (2.7), the difference equation

$$R\left(C+\frac{S\epsilon_0}{d}\right)\cdot\frac{V(t+\Delta t)-V(t)}{\Delta t}=U(t)-V(t)-RSJ(t), \tag{4.6}$$

where

$$J(t)=\frac{\Delta x}{d}\sum_{k=0}^{m-1}\frac{1}{2}\left[j(k\Delta x,t)+j((k+1)\Delta x,t)\right],$$

the stability condition for this scheme in the sense of Rutishauser [13] indeed is

$$\Delta t\leq 2[R(C+S\epsilon_0/d)],$$

as long as the dependence of J on V is small enough to be disregarded. As soon as this dependence becomes important, we can no longer consider the outer circuit separately from what happens inside. Then we will have to treat the system as a whole. Even in the case where E and α are independent of x, i.e., when space charge effects can be neglected, this leads to a nonlinear problem because of the product αj_- (α depends on V).

4.2. Difference Equation for the Electrical Field E

The differential equation (2.3) is to be solved with the side condition (2.5) imposed on the integral over the unknown function E. The most reasonable way seems to be to replace eq (2.3) by

$$\frac{E(x+\Delta xt)-E(x,t)}{\Delta x}=\epsilon_0^{-1}\tfrac{1}{2}\left[q(x+\Delta x,t)+q(x,t)\right] \tag{4.7}$$

for $x=0, \Delta x, 2\Delta x, \ldots, (M-1)\Delta x$.

Using a trapezoidal rule on the left side of eq (2.5) would lead to a side condition

$$V(t)=\sum_{m=0}^{M-1}\left\{\frac{\Delta x}{2}\left(E((m+1)\Delta x,t)+E(m\Delta x,t)\right)\right\}. \tag{4.8}$$

Equations (4.7) and (4.8) form $M+1$ equations for the $M+1$ unknowns $E(m\Delta x,t)$, $m=0, 1, \ldots, M$.

But it turns out that these equations do not give the rigorous solutions even if $q(x)$ is a linear function of x. For, take $\epsilon_0^{-1}q(x)=x$ and $V=0$, then, according to eqs (2.3) and (2.5), $E=\frac{x^2}{2}-\frac{d^2}{6}$. But, eq (4.7) yields $E=\frac{x^2}{2}+c$, where c is a constant to be determined from eq (4.8):

$$0=\frac{d^3}{6}+\frac{d(\Delta x)^2}{12}+cd.$$

Hence, we obtain $E=\frac{x^2}{2}-\frac{d^2}{6}-\frac{(\Delta x)^2}{12}$, which is not in agreement with the rigorous solution.

By partial integration on the right hand side of eq (2.5) one gets

$$V=d\cdot E(0,t)+\epsilon_0^{-1}\int_0^d (d-x)q(x,t)dx.$$

If integrated by the trapezoidal rule this formula does not give the rigorous solution for linear q either. For the above example we obtain $E=\frac{x^2}{2}-\frac{d^2}{6}+\frac{(\Delta x)^2}{6}$. Therefore, we have to search for a different method of replacing eqs (2.3) and (2.5) by finite equations such that the results are correct at least for piecewise-linear functions q. For instance, one may expect that a weighted mean of the two formulas discussed might eliminate the term containing $(\Delta x)^2$. Indeed, this way is successful if the weight ratio is 2:1. The results are then correct even for piecewise-linear functions q, as is shown below.

Let us put $\epsilon_0^{-1}q=r$ for simplicity. A piecewise-linear function $r(x)$ is given by its values at the meshpoints $r_m=r(m\Delta x)$ for $m=0, 1, \ldots, M$:

$$r(x)=r_{m-1}\cdot\frac{m\Delta x-x}{\Delta x}+r_m\frac{x-(m-1)\Delta x}{\Delta x}$$

for $(m-1)\Delta x\leq x\leq m\Delta x \quad (m=1,2,\ldots,M)$.

Our task is to find a proper approximation for the integral

$$I=\int_0^d\int_0^x r(x')dx'dx=\int_0^d(d-x)r(x)dx. \tag{4.9}$$

For the sake of simplicity we take $d=1$. The value of I follows by summation

$$\int_{\frac{(m-1)}{M}}^{m/M}(1-x)[r_{m-1}(m-Mx)+r_m(Mx-(m-1))]dx$$

$$=r_{m-1}\left[\left(1-\frac{m-1}{M}\right)\frac{1}{2M}-\frac{1}{6M^2}\right]+r_m\left[\left(1-\frac{m}{M}\right)\frac{1}{2M}+\frac{1}{6M^2}\right],$$

namely

$$I=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}+\frac{r_M-r_0}{6M^2}.$$

The trapezoidal rule applied to the rightmost expression in eq (4.9) gives the approximation

$$I_{\mathrm{tr}}=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}.$$

Repeated application of the trapezoidal rule to the middle expression in eq (4.9) gives the different approximation

$$I_{\mathrm{rep}}=\sum_{m=0}^{M-1}\frac{1}{2M}\left[\sum_{k=0}^{m-1}\frac{1}{2M}(r_k+r_{k+1})+\sum_{k=0}^{m}\frac{1}{2M}(r_k+r_{k+1})\right]=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}+\frac{r_M-r_0}{4M^2}.$$

Hence

$$1/3I_{\mathrm{tr}}+2/3I_{\mathrm{rep}}=\left[\sum_{m=0}^{M}r_m\left(1-\frac{m}{M}\right)-\frac{r_0}{2}+\frac{r_M-r_0}{6M}\right]\frac{1}{M} \tag{4.10}$$

is an approximation to eq (4.9) and leads to rigorous results in the case of piecewise linear functions $r(x)$ with slopes changing at $x=0, \Delta x, 2\Delta x, \ldots, M\Delta x=d=1$.

Since the solution of eq (4.7) requires the same summation as the inner sum in I_{rep}, we actually compute the finite approximation by repeated application of the trapezoidal rule and by adding a certain correction:

$$\frac{1}{3}I_{\mathrm{tr}}+\frac{2}{3}I_{\mathrm{rep}}=I_{\mathrm{rep}}-\frac{r_M-r_0}{12M^2}.$$

For general d, we have to replace r by rd^2.

The computation of E from V and the given values of q at the meshpoints can therefore be done by the following formulas:

At first we compute the auxiliary quantities

$$\left.\begin{aligned}&E^*(m\Delta x,t)=\frac{\Delta x}{2\epsilon_0}\sum_{k=1}^{m}(q(k\Delta x,t)+q(k-1)\Delta x,t) \qquad \text{for } m=1,2,\ldots,M.\\&E^*(0,t)=0.\end{aligned}\right\} \tag{4.11}$$

Then we continue with

$$E(0,t)=\frac{1}{d}\left[V(t)-\frac{\Delta x}{2}\sum_{m=1}^{M}(E^*(m\Delta x,t)+E^*((m-1)\Delta x,t))+(\Delta x)^2\frac{q(M\Delta x,t)-q(0,t)}{12\epsilon_0}\right] \quad (4.12)$$

$$E(m\Delta x,t)=E^*(m\Delta x,t)+E(0,t) \qquad \text{for } m=1,2,\ldots,M. \quad (4.13)$$

One may ask whether using $E(0,t)$ instead of $V(t)$ as the parameter describing the outer circuit would not simplify the computation formulas. Indeed this is true. One may compute $E(0,t)$ from a difference approximation of eq (2.8), then compute $E(m\Delta x,t)$ for $m=1,2,\ldots,M$ from eq (4.7) successively, and finally one computes V from those values by the trapezoidal rule. But it turns out that the truncation error in V is much larger in this case than if we use eqs (4.6), (4.11), (4.12), and (4.13). After a certain number of integration steps, V may even exceed U which is physically impossible, when $U=$const. and C is uncharged at time $t=0$.

4.3. Method of Computation

If we use the trapezoidal rule in eq (2.10), this boundary condition takes the form

$$j_-(0,t)=j_p+\gamma_i\cdot j_+(0,t)+\gamma_p\sum_{m=1}^{M}\left[\frac{\Delta x}{2}\left[\sigma(m\Delta x,t)j_-(m\Delta x,t)+\sigma((m-1)\Delta x,t)j_-((m-1)\Delta x,t)\right]\right]. \quad (4.14)$$

The eqs (4.1), (4.2), (4.6), (4.11), (4.12), (4.13), (2.9), and (4.14) together with initial distributions of q_+ and q_- and the initial voltage V determine the finite problem completely, provided the velocities v_+ and v_- and the ionization rates α and σ are given functions of E. Instead of the eq (4.1) one may use eq (4.4) if the electron density is in a quasi-equilibrium state.

The above system of finite equations is not completely explicit. In the case where eq (4.1) is used, explicit formulas are achieved by taking some quantities of minor importance at an earlier time. Thus, in eq (4.14) the last sum has been taken at the time $t-\Delta t$ instead of t, and in eqs (4.11) and (4.12), $q_-(0,t)$ has been replaced by $q_-(0,t-\Delta t)$. These two changes were sufficient to achieve explicit formulas for all quantities.

In the electron quasi-equilibrium case the equations are even more implicit. Therefore one uses an iterative scheme in order to obtain a solution starting from values of q_- at the previous timestep as initial approximations. The only quantity not treated iteratively is the last sum in eq (4.14). It is taken from the previous timestep throughout, i.e., instead of eq (4.14) one always uses:

$$j_-(0,t+\Delta t)=j_p+\gamma_i j_+(0,t+\Delta t)+\gamma_p\sum_{m=1}^{M}\frac{\Delta x}{2}\left[\sigma(m\Delta x,t)j_-(m\Delta x,t)+\sigma((m-1)\Delta x,t)j_-((m-1)\Delta x,t)\right]. \quad (4.15)$$

4.4. Convergence of the Iteration

It will be shown below that the proposed iteration scheme leads to convergent sequences for all quantities involved, if the electron density is small enough in the sense that it causes no distortions of the electric field comparable with the field itself.

For the sake of simplicity we consider the limit case $M\to\infty$ only. The quantities changing during the iteration are $q_-(x)$, $E(x)$, $v_-(x)$, $\alpha(x)$ and $j_-(x)$ for $0\le x\le d$, and furthermore, $v_+(0)$ and $j_+(0)$. All other quantities are fixed throughout the iteration. Since $q_-(x)$ is the only result of a previous iteration which enters the following iteration step, and since all other quantities depend on $q_-(x)$ continuously, it is sufficient to prove that the sequence of functions $q_-(x)$ converges.

According to [7], one has to consider the change $\delta q_-^{\text{new}}(x)$ of the result of one iteration step, caused by a certain change $\delta q_-^{\text{old}}(x)$ of the initial approximation. When measured by a

certain norm, the ratio of the changes has to be smaller than unity in a certain neighborhood of the true solution $q_-(x)$.

The following formulas describe the connection between $\delta q_-^{\text{new}}(x)$ and $\delta q_-^{\text{old}}(x)$. They are derived from the continuous analogs of the eqs (4.4), (4.11), (4.12), (4.13), from (4.15) and from the relations between v_-, v_+, α and E, and those between $q_\pm$, $v_\pm$ and $j_\pm$.

$$\delta q_-^{\text{new}}(x) = q_-^{\text{new}}(x)\left[\frac{\delta j_-(0)}{j_-(0)} - \frac{\delta v_-(x)}{v_-(x)} + \int_0^x \delta\alpha(u)du\right]$$

$$\frac{\delta j_-(0)}{j_-(0)} = \left(\frac{1}{v_+}\frac{dv_+}{dE}\right)\Big|_{x=0} \cdot \left[1 - \frac{j_p+\Gamma}{j_-(0)}\right]\delta E(0)$$

and Γ is the last term on the right-hand side of eq (2.10).

$$\frac{\delta v_-(x)}{v_-(x)} = \frac{1}{v_-}\frac{dv_-}{dE}\,\delta E(x)$$

$$\delta\alpha(x) = \frac{d\alpha}{dE}dE(x)$$

$$\delta E(x) = \epsilon_0^{-1}\int_0^d K(x,u)\delta q_-^{\text{old}}(u)du$$

where

$$K(x,u) = \begin{cases} u/d & \text{if } u<x \\ u/d-1 & \text{if } u>x. \end{cases}$$

We restrict ourselves to the case for which not only v_- but also v_+ is proportional to E. Then

$$\frac{1}{v_+}\frac{dv_+}{dE} = \frac{1}{v_-}\frac{dv_-}{dE} = \frac{1}{E}.$$

Furthermore, we write

$$\frac{d\alpha}{dE} = \frac{\alpha}{E}\frac{d(\ln\alpha)}{d(\ln E)}$$

since the latter differential quotient varies more slowly in the interesting range of E. We introduce the norm

$$||f|| = \int_0^d |f(x)|dx$$

for any function $f(x)$. Then, the following estimates can easily be derived:

$$|\delta E(x)| \leqq \epsilon_0^{-1}\max\left\{\frac{x}{d}, 1-\frac{x}{d}\right\}\cdot||\delta q_-^{\text{old}}||$$

$$||\delta q_-^{\text{new}}|| \leqq K\frac{\epsilon_0^{-1}\int_0^d |q_-^{\text{new}}(x)|dx}{\min_x |E(x)|}\cdot||\delta q_-^{\text{old}}||$$

where

$$K = 1 + \frac{\min_x |E(x)|}{E(0)} + \frac{3}{4}\alpha d\max_x\left|\frac{d(\ln\alpha)}{d(\ln E)}\right|.$$

The condition for convergence is that

$$K\frac{\epsilon_0^{-1}\int_0^d |q_-^{\text{new}}(x)|dx}{\min_x |E(x)|} = Q < 1 \tag{4.16}$$

which can be interpreted the following way: The field distortions produced by the electrons and amplified by the numerical factor K must not exceed the minimum field strength in the gap.

From the rate of convergence given by eq (4.16) and the change of $q_-(x)$ between successive iterations, the corrections introduced by a further iterative step can be estimated. According to these considerations, the iteration may be stopped after a prescribed accuracy has been reached. A criterion of this kind has been used in the code.

5. Formal Solutions for Constant Field

5.1. General Considerations and Formulas

The nonlinearities contained in the equations of section 2 and in the corresponding difference equations disappear as soon as the quantities α, σ, v_+ v_- can be regarded as independent of the solution. If, moreover, these quantities are constant in space and time and if the voltage V across the gap is constant, the equations of section 2 and the corresponding difference equations can be solved explicitly. Then, of course, there is no room for eq (2.3), i.e., this treatment disregards space charge, and eq (2.7) cannot be taken into account, i.e., the reactions in the outer circuit are disregarded, or, in other words, the external resistance R is so small that it can be neglected by putting $R=0$, which leads to $U=V$. Work in this direction has been done by several authors [1, 2, 3, 9] as far as the differential equations are concerned. These papers discuss what happens in the tube. Here this special case will be considered again, but for a different purpose. We shall discuss the difference equations along with the differential equations in order to get insight into the effects of truncation errors. We hope that, to a certain extent, these effects carry over to the more general case, and therefore will allow us to correct the results obtained with finite steplengths Δt and Δx so that we obtain closer approximations to the case of infinitesimal steplengths.

We mainly deal with the difference equations. The results for the differential equations will be obtained by letting Δt, $\Delta x \to 0$. We restate the equations of section 4 for our special case putting $t_0=0$:

$$\frac{j_-(x, t+\Delta t)-j_-(x, t)}{v_-\Delta t}=j_-(x, t)-\frac{j_-(x, t)-j_-(x-\Delta x, t)}{\Delta x} \tag{5.1a}$$

for $x=\Delta x, 2\Delta x, \ldots M\Delta x=d$; $t=0, \Delta t, 2\Delta t, \ldots$

In the electron quasi-equilibrium case, we shall use instead of this:

$$j_-(x, t)=j_-(0, t)\cdot e^{\alpha x} \tag{5.1b}$$

for the same x and t as above. The ion currents behave according to

$$\frac{j_+(x, t+\Delta t)-j_+(x, t)}{v_+\Delta t}=j_-(x, t)+\frac{j_+(x+\Delta x, t)-j_+(x, t)}{\Delta x} \tag{5.2}$$

for $x=0, \Delta x, \ldots (M-1)\Delta x$; $t=0, \Delta t, 2\Delta t, \ldots$

There are two boundary conditions, namely

$$j_+(d, t)=0 \qquad \text{for } t=0, \Delta t, 2\Delta t, \ldots \tag{5.3}$$

and

$$j_-(0, t)=j_p+\gamma_i j_+(0, t)+\gamma_p \sum_{m=1}^{M} \frac{\Delta x}{2}\,\sigma\cdot[j_-(m\Delta x, t-\Delta t)+j_-((m-1)\Delta x, t-\Delta t)]$$
$$\text{for } t=\Delta t, 2\Delta t, \ldots \tag{5.4}$$

In order to make the solution unique, we have to introduce initial conditions, for instance by prescribing $j_-(x,0)$ and $j_+(x, 0)$ for $x=0, \Delta x, \ldots M\Delta x$. But we will be mainly concerned with

solutions which are proportional to $e^{\lambda t}$ with a suitable λ. For this type of solution, initial conditions do not have to be given.

The above equations are linear and homogeneous with the single exception of (5.4) which contains the inhomogeneous term j_p. Since j_p is time independent and since time does not appear explicitly in any other term, there will exist, at least in general, a time independent solution. More general solutions will be obtained by superposition of any solutions of the corresponding homogeneous system with $j_p=0$. The homogeneous system allows for time separation in the form of a factor $e^{\lambda t}$. Then the stationary case results from putting $\lambda=0$ and from slight c' anges due to the inhomogeneous term. We introduce $j_\lambda^+(x)$ and $j_\lambda^-(x)$ by

$$\begin{aligned} j_-(x,t) &= j_\lambda^-(x)\cdot e^{\lambda t} \\ j_+(x,t) &= j_\lambda^+(x)\cdot e^{\lambda t} \end{aligned} \tag{5.5}$$

Then, eq (5.1a) can be solved by

$$j_\lambda^-(x)=j_\lambda^-(0)\cdot e^{\alpha^*\Delta x}, \tag{5.6}$$

where α^* is the solution of the transcendental equation

$$\frac{e^{\lambda\Delta t}-1}{v_-\cdot\Delta t}=\alpha-\frac{1-e^{-\alpha^*\Delta x}}{\Delta x} \tag{5.7}$$

In the electron quasi-equilibrium case, i.e., if we use eq (5.1b), all results will be correct in the sequel if we replace α^* by α, unless we distinguish explicitly between the two cases. Equation (5.2) now reduces to a single inhomogeneous equation for $j_\lambda^+(x)$, the inhomogeneous term being $\alpha j_\lambda^-(x)$. The solution satisfying the boundary condition (5.3) is easily found to be

$$j_\lambda^+(x)=Ae^{\beta x}[e^{(\alpha^*-\beta)d}-e^{(\alpha^*-\beta)x}], \tag{5.8}$$

where

$$A=\frac{j_\lambda^-(0)}{\dfrac{e^{\alpha^*\Delta x}-1}{\alpha\Delta x}-\dfrac{e^{\lambda\Delta t}-1}{v_+\alpha\Delta t}}$$

and β is determined by the equation

$$\frac{e^{\lambda\Delta t}-1}{v_+\Delta t}=\frac{e^{\beta\Delta x}-1}{\Delta x}. \tag{5.9}$$

The remaining unknowns are $j_\lambda^-(0)$ and λ. In the homogeneous case the quantity $j_\lambda^-(0)$ remains free, whereas λ follows from eq (5.4) with $j_p=0$, which transforms into the transcendental equation

$$1=\frac{\gamma_i[e^{(\alpha^*-\beta)d}-1]}{\dfrac{e^{\alpha^*\Delta x}-1}{\alpha\Delta x}-\dfrac{e^{\lambda\Delta t}-1}{v_+\alpha\Delta t}}+\gamma_p\sigma\Delta x e^{-\lambda\Delta t}\cdot\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\alpha^*d}-1}{e^{\alpha^*\Delta x}-1}.$$

5.2. Stationary Solution

For the stationary solution of the inhomogeneous eq (5.4), we put $\lambda=0$. Then (5.9) leads to $\beta=0$, and (5.4) shows that

$$j_0^-(0)=j_p\Big/\left\{1-\frac{\alpha\Delta x(e^{\alpha^*d}-1)}{e^{\alpha^*\Delta x}-1}\cdot\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\frac{1+e^{\alpha^*\Delta x}}{2}\right)\right\}. \tag{5.11}$$

If we use (5.1a), we may simplify our equations further. From eq (5.7) we obtain α^* explicitly:

$$\alpha^*=\frac{1}{\Delta x}\ln\frac{1}{1-\alpha\Delta x}. \tag{5.12}$$

Equation (5.8) reduces to

$$j_0^+(x)=j_0^-(0)\cdot(1-\alpha\Delta x)(e^{\alpha^* d}-e^{\alpha^* x}), \tag{5.13}$$

and eq (5.11) reduces to

$$j_0^-(0)=j_p\Big/\left\{1-[e^{\alpha^* d}-1]\left[\gamma_i(1-\alpha\Delta x)+\gamma_p\frac{\sigma}{\alpha}\left(1-\frac{1}{2}\alpha\Delta x\right)\right]\right\}. \tag{5.14}$$

In the electron quasi-equilibrium case, we use eq (5.1b) and obtain

$$j_0^+(x)=j_0^-(0)\cdot\frac{\alpha\Delta x}{e^{\alpha\Delta x}-1}\,(e^{\alpha d}-e^{\alpha x}) \tag{5.15}$$

and

$$j_0^-(0)=j_p\Big/\left\{1-\frac{\alpha\Delta x}{e^{\alpha\Delta x}-1}\,(e^{\alpha_i}-1)\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\frac{1+e^{\alpha\Delta x}}{2}\right)\right\}. \tag{5.16}$$

By letting $\Delta x\to 0$ we obtain the corresponding relations for the differential equations; namely $\alpha^*=\alpha$ and, furthermore,

$$j_0^+(x)=j_0^-(0)\cdot(e^{\alpha d}-e^{\alpha x}), \tag{5.17}$$

$$j_0^-(0)=j_p\Big/\left\{1-(e^{\alpha d}-1)\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\right)\right\}. \tag{5.18}$$

When the respective denominators in eqs (5.14), (5.16), (5.18), become zero, no stationary solution is possible. The voltage V for which this occurs is called the breakdown voltage. If we assume that σ/α is a given constant independent of V (it is often assumed that $\sigma=\alpha$) then the three respective denominators are equal to 1 for $\alpha=0$ and decrease monotonically in the interval $0\leq\alpha<\infty$. Therefore, there exists one and only one positive real root α, corresponding to the breakdown voltage. If the voltage V exceeds this value, the theoretical stationary solution shows negative values of j^-, which is physically impossible: No stable solutions exist beyond the breakdown voltage.

The physically interesting things happen if V is near the breakdown voltage. Therefore we have to look into the dependence of the breakdown voltage on the steplength Δx in order to establish a base for comparison of numerically obtained results and the true theoretical solutions, or the experimental measurements in this case.

In the infinitesimal case the value $\alpha=\alpha_{\rm br}$ corresponding to breakdown can be expressed explicitly from (5.18), if σ/α is given:

$$\alpha_{\rm br}=\frac{1}{d}\ln\left(\frac{1}{\gamma_i+\gamma_p(\sigma/\alpha)_{\rm br}}+1\right).$$

For the difference schemes, the corresponding transformation will lead to a form of the equation suitable for iteration. From eq (5.16), i.e., for the electron quasi-equilibrium case, we get

$$\alpha_{\rm br}=\frac{1}{d}\ln\left[1+\frac{\exp(\alpha_{\rm br}\Delta x)-1}{\alpha_{\rm br}\Delta x}\Big/\left(\gamma_i+\gamma_p\left(\frac{\sigma}{\alpha}\right)_{\rm br}\frac{1+\exp(\alpha_{\rm br}\Delta x)}{2}\right)\right].$$

In case we use eq (5.1a) instead, it follows from eqs (5.14) or (5.11) that

$$\alpha^*_{\rm br}=\frac{1}{d}\ln\left[1+\exp(\alpha^*_{\rm br}\Delta x)\Big/\left(\gamma_i+\gamma_p\left(\frac{\sigma}{\alpha}\right)_{\rm br}\frac{1+\exp(\alpha^*_{\rm br}\Delta x)}{2}\right)\right].$$

We shall see that these formulas are special cases of more general formulas for the homogeneous system.

5.3. Special Time-Dependent Solutions

We turn now to a discussion of the homogeneous system, especially the transcendental eq (5.10). Let us first consider the infinitesimal analog of this equation, obtained by letting $\Delta x \to 0$ and $\Delta t \to 0$. Equation (5.9) becomes then

$$\beta = \lambda / v_+, \tag{5.19}$$

and eq (5.7) transforms into

$$\alpha^* = \alpha - \lambda / v_-. \tag{5.20}$$

If we define v by $1/v = 1/v_+ + 1/v_-$, then eq (5.10) becomes

$$1 = \frac{\gamma_f \alpha}{\alpha - \lambda/v} \left[e^{(\alpha - \lambda/v)d} - 1 \right] + \frac{\gamma_p \sigma}{\alpha - \lambda/v_-} \left[e^{(\alpha - \lambda/v_-)d} - 1 \right], \tag{5.21}$$

in accordance with [9]. For the breakdown voltage, $\lambda = 0$ is a solution.

For voltages near the breakdown voltage, there must be a solution λ near zero for continuity reasons. For this special root, the equation may be transformed into an iterative scheme similar to the one for the computation of breakdown voltage itself. We show this for the case where a solution is wanted for λ as a function of α.
We put

$$\alpha - \lambda/v = \alpha(1-u)$$

and rewrite eq (5.21):

$$1 = \left[e^{\alpha(1-u)d} - 1 \right] \cdot \left[\frac{\gamma_f}{1-u} + \frac{\gamma_p(\sigma/\alpha)}{1 - \lambda/(\alpha v_-)} \cdot \frac{e^{\lambda d/v_+} - e^{-\alpha(1-u)d}}{1 - e^{-\alpha(1-u)d}} \right].$$

According to the assumptions made above, only the first factor on the right changes rapidly with α. Therefore we solve the equation for this factor, obtaining

$$\alpha = \frac{1}{(1-u)d} \ln \left\{ 1 + 1 \Big/ \left(\frac{\gamma_f}{1-u} + \frac{\gamma_p(\sigma/\alpha)}{1 - u \cdot v/v_-} \cdot \frac{e^{\lambda d/v_+} - e^{-\alpha(1-u)d}}{1 - e^{-\alpha(1-u)d}} \right) \right\}. \tag{5.22}$$

This form is suitable for iteration since the right hand side is only slowly varying with α, particularly, if we use u as an independent parameter. We describe the function $\lambda(\alpha)$ by the parametric representation $\lambda = \lambda(u)$, $\alpha = \alpha(u)$. From a first approximation of V and α we compute v_+, v_-, v, σ, and

$$\lambda = \alpha v u. \tag{5.23}$$

Since all variables on the right hand side of eq (5.22) are known, a new approximation for α can be computed from eq (5.22). Since the right hand side is slowly varying with α for sufficiently small u, we can hope for fast convergence of the iteration.

A similar but slightly more complicated approach is possible for solving eq (5.10), if eq (5.1a) is used for the computation of electron current. Here, in addition, the relation between α^* and α has to be established iteratively. Let us assume that we know a first approximation for α^* for a given value of the independent parameter

$$u = \frac{e^{\lambda \Delta t} - 1}{\alpha v \Delta t}.$$

Then α can be computed from (5.7) if we assume in addition that v/v_- is at least approximately known:

$$\alpha=\frac{1-e^{-\alpha^*\Delta x}}{\Delta x}\Big/\left(1-\frac{uv}{v_-}\right). \tag{5.24}$$

If necessary, from the α thus obtained one may compute a new value of v/v_- and repeat the computation of α iteratively.

From u and α one may compute

$$\lambda=\frac{1}{\Delta t}\ln\left(1+\alpha v u\Delta t\right) \tag{5.25}$$

and

$$\beta=\frac{1}{\Delta x}\ln\left(1+\alpha\frac{v}{v_+}u\Delta x\right). \tag{5.26}$$

Equation (5.10) may be written as

$$1=[e^{(\alpha^*-\beta)d}-1]\cdot\left[\frac{\gamma_i}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{-\lambda\Delta t}}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)}\cdot\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\beta d}-e^{-(\alpha^*-\beta)d}}{1-e^{-(\alpha^*-\beta)d}}\right]$$

and is solved for the first factor:

$$\alpha^*=\frac{1}{(1-\beta/\alpha^*)d}\ln\left[1+\left(\frac{\gamma_i}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{-\lambda\Delta t}}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)}\cdot\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\beta d}-e^{-(\alpha^*-\beta)d}}{1-e^{-(\alpha^*-\beta)d}}\right)^{-1}\right] \tag{5.27}$$

The computation of the breakdown voltage thus appears as a special case of these formulas, namely as the case $u=\lambda=0$.

In the equilibrium case the formula corresponding to eq (5.27) is

$$\alpha=\frac{1}{(1-\beta/\alpha)d}\ln\left[1+\left(\frac{\gamma_i}{(e^{\alpha\Delta x}-1)/(\alpha\Delta x)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{\lambda\Delta t}}{(e^{\alpha\Delta x}-1)/(\alpha\Delta x)}\cdot\frac{1+e^{\alpha\Delta x}}{2}\frac{e^{\beta d}-e^{-(\alpha-\beta)d}}{1-e^{-(\alpha-\beta)d}}\right)^{-1}\right]. \tag{5.28}$$

This equation is used together with eqs (5.25) and (5.26) for solving eq (5.10) iteratively.

Because of the similarities between the eqs (5.22), (5.27), and (5.28) one can treat all three of these equations quite simply in a single computer program.

5.4 General Time-Dependent Solution for Constant Field

So far we have considered only a special root of eq (5.10). However, one can see that a transcendental equation of the type of eq (5.10) and its limit case eq (5.21) has more roots λ in general. For the case of finite Δt and Δx, the equation is rational in $e^{\lambda\Delta t}$. Therefore, there is a finite number of roots $e^{\lambda\Delta t}$, each of which corresponds to a infinite set of roots λ of the form $\lambda=\lambda_0+2\pi ik/\Delta t$ ($k=0, \pm1, \pm2, \ldots$). But all these roots describe the same function on the grid. Equation (5.21), however, has an infinity of roots λ, whose asymptotic distribution is shown in [6]. It turns out that at most a finite number of them can have a real part larger than the real root discussed before. The root with the largest real part will become dominant in any solution as time goes on. Therefore, one is mainly interested in the root with the largest real part. If we knew that this root was the real root discussed in section 5.3, we could confine our considerations to this root essentially. Unfortunately, a proof is not available at the time being. Therefore, it remains an open question whether or not the asymptotic behavior of the solution for larger t can be described by the formulas given in section 5.3.

For a complete formal solution of an arbitrary initial value problem, of course, one has to consider all of the solutions of eqs (5.10) and (5.21) and one has to develop the initial distribution into a series of the corresponding functions after substracting the stationary solution of the nonhomogeneous equation. No attempt has been made to go further in this direction, but see [1] for some results of this kind.

6. Results

The physical significance of the results obtained by the machine computations is discussed in [5, 11]. Therefore, we restrict our discussion to the more mathematical questions. The main question is how large the truncation error is, i.e., the error introduced by using finite differences instead of derivatives. No attempts to establish rigorous error bounds have been made. Instead of this, experiments with different step lengths have been carried out for the following set of parameters:

$j_p=10^{-12}$ amp/cm^2, $p=722$ torr, $d=1$ cm, $S=1$ cm^2,
$C=10^{-11}$ amp sec/volt, $R=10^6$ volt/amp, $U=V=25.6$ kv for $t\leq 0$.

U for $t>0$, γ_i, γ_p and Δx are different for different curves. The time step Δt is given by the formula $\Delta t=0.8\cdot\Delta x/\max_x\{v_+\}$ in the electron quasi-equilibrium case, $\Delta t=0.8\cdot\Delta x/\max_x\{v_-\}$ in the general case, unless something else is stated explicitly. The constants used to describe the functions α and σ are $C_1=C_2=8$ cm^{-1} torr^{-1}, $D_1=D_2=247$ volt cm^{-1} torr^{-1}, the mobilities of electrons and ions are constant: $\mu_-=4\times10^5$ torr cm^2/(sec volt), $\mu_+=2\times10^3$ torr cm^2/(sec volt), $B_1=0$, $W_1=+\infty$.

The calculations in section 5 show that in the most interesting area near the breakdown voltage the solution is very sensitive to changes in the voltage. On the other hand, a finite stepwidth of reasonable size, e.g., with 20 subintervals, introduces a change of the breakdown voltage of notable magnitude. Therefore, the differences between runs with different stepwidths are mainly due to the influence of the stepwidth on the breakdown voltage. It seems to be feasible to eliminate this influence by relating the applied voltages to the breakdown voltage, as computed according to section 5.3, i.e., neglecting space charge, for the stepwidth used in each case. This method turned out to give very satisfactory results for the electron quasi-equilibrium case, as can be seen from figures 2 and 3.

The reason why even the relatively large steplength $\Delta x=0.05$ cm ($\alpha\Delta x\approx0.5$ to 0.6) gives a good approximation, can be seen from the figures 4, 5, and 6, where the coefficient of temporal growth λ, as computed from the formulas of section 5.3, i.e., without space charge effects, is plotted versus the voltage V across the gap and the overvoltage $V-V_{br}$. Figure 4 shows that the curves for the difference equation (quasi-equilibrium case, $\Delta x=0.05$ cm) and for the differential equation go almost parallel over a long range. Therefore after relating the voltages to the breakdown voltage, the curves almost coincide as can be seen from figure 5. This explains the good results obtained with that approximation, at least as long as space charge effects have small influence.

With the same value of $\gamma_i+\gamma_p$, but a portion of 10 and 20 percent γ_p, the temporal development goes faster than for $\gamma_p=0$, as one could expect from physical considerations.

According to figure 6, the difference scheme should give an approximation almost as good as for $\gamma_p=0$, even a better one in the 10 percent case, where the curves for the differential equation and the difference solution nearly coincide.

The influence of the time lag in the term describing the production of electrons by secondary photons (last term in eq (5.10)) has been studied by introducing an artificial factor $\exp(-\lambda\Delta t)$ with γ_p into eq (5.22) for the continuous case. Δt was assigned a fixed value approximately equal to the ones used in the computation with $\Delta x=0.025$, namely $\Delta t=6.25\times10^{-7}$ sec. The deviations due to that factor $\exp(-\lambda\Delta t)$ can be seen from table 1 below.

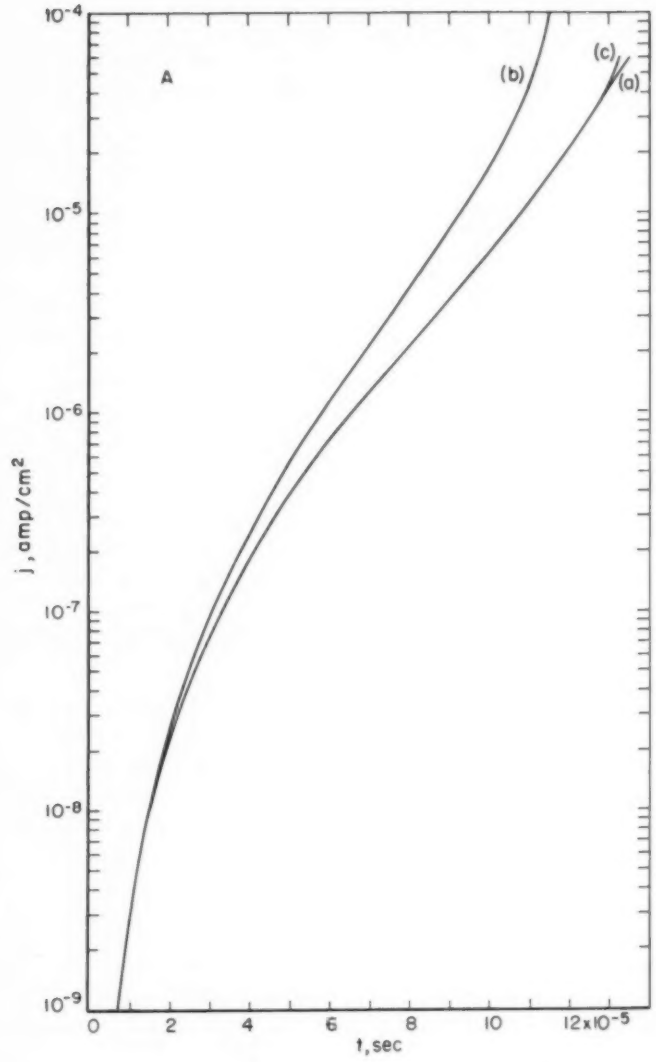

FIGURE 2A. *Mean current density* J *across the gap versus time* t *for* $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$.

	Δx[cm]	U[kv]	$U-V_{br}$[kv]
(a)	0.05	28.870	0.233
(b)	.025	28.870	.295
(c)	.025	28.807	.233

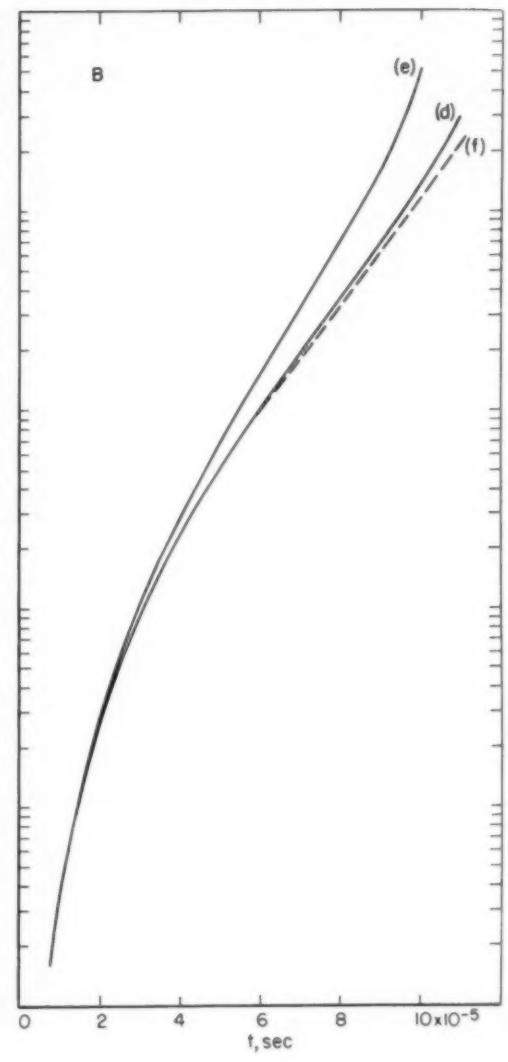

FIGURE 2B. *Mean current density* J *across the gap versus time* t *for* $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$.

	x[cm]	U[kv]	$U-V_{br}$[kv]
(d)	0.05	28.854	0.233
(e)	.025	28.854	.287
(f)	.025	28.800	.233

We conclude that the influence of that time lag is not very important, at least for the overvoltages and the small rates γ_p of production of secondary electrons considered here. We see that the time lag slows down the speed of development by a few percent at most, even in the worst case.

The electron quasi-equilibrium assumption $\partial q_-/\partial t=0$ in (2.1) is equivalent to letting $v_-\to\infty$ in this equation. Therefore, the influence of that assumption in the differential equation may be studied by replacing v_- by ∞ and v by v_+ in the eqs (5.22) and (5.23). If $\gamma_p=0$, the result is obvious. For fixed voltage, i.e., fixed α, the introduction of that assumption

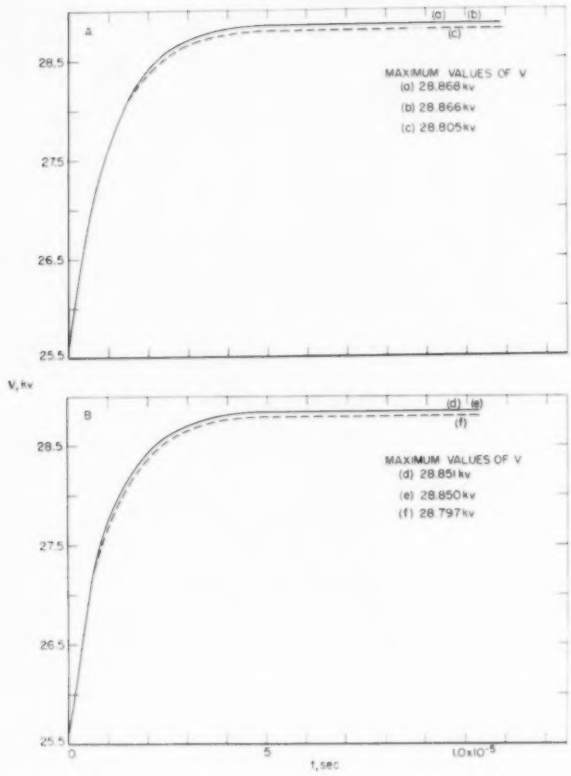

FIGURE 3A. *Voltage* V *across the gap versus time* t *for parameters of figure 2A.*

FIGURE 3B *Voltage* V *across the gap versus time* t *for parameters of figure 2B.*

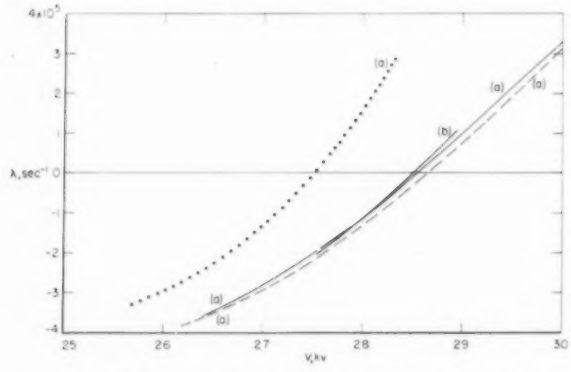

FIGURE 4. *Coefficient* λ *of temporal growth for constant field versus voltage* V *across the gap, for* Δx=0.05 *cm.*

(a) $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$
(b) $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$ (continuous case only) for electron quasi-equilibrium case (dashed) general case (dotted) continuous case (solid).

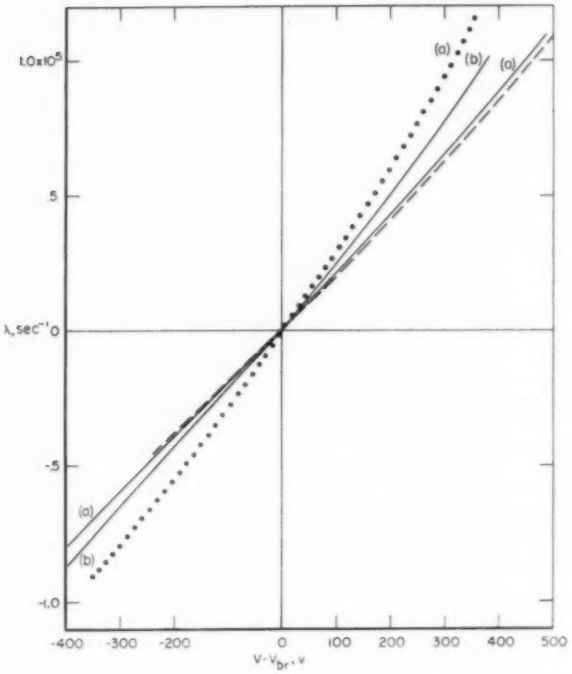

FIGURE 5. *Coefficient* λ *of temporal growth for constant field versus overvoltage* V—V_{br} *for same parameters as figure 4.*

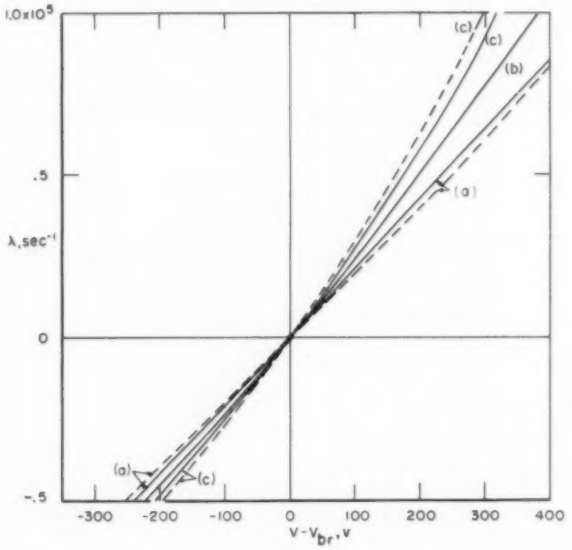

FIGURE 6. *Coefficient* λ *of temporal growth for constant field versus overvoltage* V—V_{br} *for* Δx=0.05 *cm.*

(a) $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$
(b) $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$ (continuous and quasi-equilibrium case coincide)
(c) $\gamma_p=3.0\ 10^{-6}$, $\gamma_i=1.20\ 10^{-5}$ for electron quasi-equilibrium case (dashed) continuous case (solid).

TABLE 1. *Influence of the time lag Δt for secondary photons on the coefficient λ of temporal growth*

$V-V_{br}$	$\frac{\gamma_p}{\gamma_i+\gamma_p}=0.1$		$\frac{\gamma_p}{\gamma_i+\gamma_p}=0.2$	
	$\Delta t=0$	$\Delta t=6.25\ 10^{-7}$	$\Delta t=0$	$t=6.25\ 10^{-7}$
V_{ou}				
−400	−0.8643	−0.8624	−0.9291	−0.9248
−300	−.6595	−.6576	−.7147	−.7105
−200	−.4479	−.4463	−.4901	−.4864
−100	−.2285	−.2274	−.2532	−.2506
0	0	0	0	0
+100	+0.2396	+0.2377	+0.2746	+0.2696
+200	+.4930	+.4878	+.5802	+.5651
+300	+.7605	+.7538	+.9317	+.8978
+400	+1.0610	+1.0410	+1.3650	+1.2880

amounts to no change in u, and therefore to a relative increase of λ by about v_+/v_-. Hence the temporal development is speeded up by about 0.5 percent, which is small enough for the accuracy required in this problem.

A detailed analysis of the effect of that assumption in the case $\gamma_p \neq 0$, which is elementary, but too lengthy to be reproduced here, shows that this remains true as long as the relative influence of γ_p as compared to γ_i in eq (5.22) is small, i.e., roughly speaking, as long as $\gamma_p \cdot e^{\lambda d/v_+} \ll \gamma_i$. This is in agreement with physical considerations, since, if the influence of secondary photons is dominant, the development is dominated by a process with feedback, all of whose components go infinitely fast. Quantitatively, as long as $u \ll 1$, $v_+ \ll v_-$, $\alpha d \gg 1$, the change $\delta\lambda$ of λ effected by the equilibrium assumption is, with good approximation.

$$\frac{\delta\lambda}{\lambda} = \frac{v_+}{v_-} \cdot \left(1 + \frac{\alpha d - 1}{\alpha d - 1/(1-u)} \cdot \frac{\gamma_p(\sigma/\alpha)(1-u)e^{\lambda d/v_+}}{\gamma_i}\right).$$

The full difference equations, with eq (5.1a) included, have not been used[2], for the following reasons. For the same Δx and the same time interval to be covered, the stability conditions require 200 times as many time steps, if the electron-quasi-equilibrium condition is dropped. Furthermore, figures 3 and 4 show that, for $\Delta x = 0.05$ cm, the deviation from the limit case $\Delta x = 0$ is considerable, and cannot be diminished satisfactorily by a simple change in the outer voltage leading to the same overvoltage. It is estimated that a decrease of Δx to 0.01 cm is at least necessary in order to obtain results as good as the ones in the quasi-equilibrium case. Two trial runs were made, both leading to breakdown after a few ion transit times. The data for these runs were:

(a) $\Delta x = 0.05$, $U = 31.5$ kv, $V_{br} = 28.64$ kv, $\gamma_i = 1.5\ 10^{-5}$, $\gamma_p = 0$
infinite current after 2 ion transit times.
(b) $\Delta x = 0.5$, $U = 28.87$ kv, $V_{br} = 28.51$ kv, $\gamma_i = 0$, $\gamma_p = 1.5\ 10^{-5}$
infinite current after 3 ion transit times.

All the above considerations and the computations done so far are restricted to a domain where the temporal growth in one (natural) time-step (as indicated by the stability condition) is not too large. Near the actual breakdown, therefore, these arguments may not apply, and an investigation of the nature and quantitative features of the breakdown singularity is not given in this paper. Since the time required for the space change to become significant is very large compared to the remaining time until completion of the breakdown, and since the assumption of one-dimensionality is doubted for high current densities, there was no point in trying to describe the details of the breakdown more precisely.

The features discussed in this section have been tested only for a few sets of parameters. Up to this time it has not been proved that the conclusions are general. Since the number of

[2] Several successful computer runs have been made since by Dr. A. L. Ward, using this option of the program.

parameters is quite large, it has not been attempted to explore the limits of the region where the conclusions are valid. But the methods of computing and of guessing the effect of truncation errors as described here should be tried in other cases.

The authors express their thanks to Dr. A. L. Ward of the Harry Diamond Laboratories, Department of the Army, for suggesting the problem, and to J. P. Menard and A. E. Beam, who did the computer programming.

7. References

[1] Miyoshi, Y., Theoretical analysis of buildup of current in transient Townsend discharge, Phys. Rev. **103,** 1609–1618 (1956).
[2] Miyoshi, Y., Development of space charge and growth of ionization in the transient Townsend discharge, Phys. Rev. **117,** 355–356 (1960).
[3] Auer, P. L., Transient analysis of the Townsend discharge, Phys. Rev. **111,** 671–682 (1958).
[4] Ward, A. L., Effect of space charge in cold-cathode gas discharges, Phys. Rev. **112,** 1852–1857 (1958).
[5] Ward, A. L., Electronic computation of the effect of space charge in the temporal growth of current, 5th Int'l Conf. on Ionization Phenomena in Gases, Munich 1961, North Holl. Publ. Co. (1961).
[6] Langer, R. E., The asymptotic location of the roots of a certain transcendental equation, Trans. Amer. Math. Soc. **31,** 837–844 (1929).
[7] Collatz, L., Einige Anwendungen funktionalanalytischer Methoden in der praktischen Analysis, Z. angew. Math. Phys. **4,** 327–357. (1953).
[8] Lax P. D., and Richtmyer, R. D., Survey of the stability of linear finite difference equations, Comm. Pure Appl. Math. **9,** p. 267 (1956).
[9] Menes, M., Buildup of a discharge in argon, Phys. Rev. **116,** 481–486, (1959).
[10] Ward, A. L., and Jones, E., Electrical breakdown in hydrogen at low pressures, Phys. Rev. **122,** 376–380, (1961).
[11] Ward, A. L., Calculations of cathode fall characteristics J. Appl. Phys. **33,** 2789–2794, (1962).
[12] Richtmyer, R. D., Difference methods for initial value problems. Interscience Publishers New York, London, (1957). 238 pp.
[13] Rutishauser, H. Über die Instabilität von Methoden zur Integration gewöhnlicher Differentialgleichungen. Z. angew. Math. Phys. **3,** 65–74 (1952).

(Paper 67B1–92)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

Tables of Genera of Groups of Linear Fractional Transformations*

Harriet Fell, Morris Newman, and Edward Ordman

(October 29, 1962)

The genera of the groups $\Gamma_0(n)$, $\Gamma^*(n)$ together with certain associated number-theoretic functions are given for $1 \leq n \leq 1000$.

The 2×2 modular group Γ and its subgroups are of fundamental importance in the theory of automorphic functions (in particular the elliptic modular functions) and in the theory of Riemann surfaces. Let G be such a subgroup. Then the substitutions

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{1}$$

of G map the upper τ half-plane onto itself, and the points of this half-plane are partitioned into equivalence classes modulo G. A set of points consisting of one point from each equivalence class is termed a *fundamental set*, and if G is of finite index in Γ there is a simple way of selecting a standard fundamental set R, which is fully described in Ford's book [1] [1] or in Gunning's book [3]. The set R is commonly called a *fundamental region*. After appropriate identifications of sides and vertices are made R becomes a surface whose genus g plays a central role in the study of G.

Among the subgroups of finite index in Γ the congruence subgroups have been studied most extensively and among the congruence subgroups the (nonnormal) subgroups Γ_0 (n) are of primary importance. These are defined for every natural number n as the totality of substitutions (1) where a, b, c, d are rational integers, $ad-bc=1$, and $c\equiv 0 \pmod{n}$.

If the substitution

$$\tau' = -\frac{1}{n\tau}$$

is adjoined to $\Gamma_0(n)$ the larger group so obtained (in general not a subgroup of Γ) is donoted by $\Gamma^*(n)$. Formulas for the genera $g_0(n)$, $g^*(n)$ of $\Gamma_0(n)$, $\Gamma^*(n)$, respectively, have been given by F. Klein [6], E. Hecke [4,5], and R. Fricke [2]. Denote the number of solutions of the congruence

$$x^2 - x + 1 \equiv 0 \pmod{n},\ 0 \leq x \leq n-1 \tag{2}$$

by $\nu_1(n)$, and the number of solutions of the congruence

$$x^2 + 1 \equiv 0 \pmod{n},\ 0 \leq x \leq n-1 \tag{3}$$

by $\nu_2(n)$. Set

$$\mu(n) = n \prod_{q \mid n} \left(1 + \frac{1}{q}\right),$$

$$\sigma_0(n) = \sum_{d \mid n} \varphi\left(d, \frac{n}{d}\right) = \prod_{q^e \| n} \left(q^{\left[\frac{e}{2}\right]} + q^{\left[\frac{e-1}{2}\right]}\right),$$

where $\left(d, \frac{n}{d}\right)$ denotes the greatest common divisor of d and $\frac{n}{d}$, and φ is the Euler function. Then $\nu_1(n)$ is the number of inequivalent elliptic vertices of period 3 of the fundamental region R_n of $\Gamma_0(n)$, $\nu_2(n)$ the number of inequivalent elliptic vertices of period 2 of R_n, $\mu(n)$ the index of $\Gamma_0(n)$ in Γ, and $\sigma_0(n)$ the number of inequivalent parabolic vertices of R_n. The genus $g_0(n)$ of $\Gamma_0(n)$ is then

$$g_0(n) = 1 + \frac{\mu(n)}{12} - \frac{\nu_1(n)}{3} - \frac{\nu_2(n)}{4} - \frac{\sigma_0(n)}{2}.$$

Furthermore let $h(-4n)$ denote the number of classes of primitive positive binary quadratic forms of discriminant $-4n$, and set

$$\delta_n = \begin{cases} 2 & n \equiv 7 \pmod 8 \\ \frac{4}{3} & n \equiv 3 \pmod 8, n > 3 \\ 1 & \text{otherwise.} \end{cases}$$

Then the genus g^* (n) of Γ^* (n) is given by

$$g^*(n) = \frac{1}{2}\, g_0(n) + \frac{1}{2} - \frac{1}{4}\, \delta_n h(-4n).$$

In this article we give the actual numerical values of $g_0(n)$, $g^*(n)$ for $1 \leq n \leq 1000$ together with the values of the associated functions $\nu_1(n)$, $\nu_2(n)$, $\mu(n)$, $\sigma_0(n)$, h$(-4n)$. These were computed on the IBM 7090 of the NBS in a negligible amount of time. The resulting tables are of considerable interest however and should prove useful in many number-

*The computation of the tables was carried out by the first and third authors at the suggestion of the second author, as a summer project at NBS. The work was supported by the Office of Naval Research.

[1] Figures in brackets indicate the literature references at the end of this paper.

theoretic investigations, as well as in the study of the groups $\Gamma_0(n)$, $\Gamma^*(n)$.

We say some words about the computation of the tables. The functions $\nu_1(n)$, $\nu_2(n)$ were computed directly as the number of solutions of the congruences (2), (3) respectively. It is easy to give closed expressions for $\nu_1(n)$, $\nu_2(n)$ however. These are

$$\nu_1(n)=\begin{cases}0 & 2|n \text{ or } 9|n\\ \prod_{q|n}\left(1+\left(\frac{-3}{q}\right)\right) & \text{otherwise,}\end{cases} \tag{4}$$

$$\nu_2(n)=\begin{cases}0 & 4|n\\ \prod_{q|n}\left(1+\left(\frac{-4}{q}\right)\right) & \text{otherwise.}\end{cases} \tag{5}$$

We see that (4) and (5) imply that for $n>3$, $\nu_1(n)$ and $\nu_2(n)$ can take only 0 or powers of 2 as values. This provides an excellent check on the computation of $\nu_1(n)$, $\nu_2(n)$.

The function $\mu(n)$ is a multiplicative arithmetic function of n with easily computed values at the prime powers. In addition, if q is a prime $\mu(n)$ satisfies

$$\mu(nq)=\begin{cases}q\mu(n) & q|n\\ (q+1)\mu(n) & \text{otherwise.}\end{cases} \tag{6}$$

Relationship (6) provides an efficient and easily applied check on the computation of $\mu(n)$. Similar remarks apply to the function $\sigma_0(n)$.

The class number $h(-4n)$ was computed as the number of solutions of the system

$$\begin{cases}b^2-4ac=-4n\\ (a,b,c)=1\\ -a<b\leq a<c \text{ or } 0\leq b\leq a=c.\end{cases} \tag{7}$$

A check on the computation was provided by determining the parity of $h(-4n)$. It is not difficult to show from (7) that

$$h(-4n)\equiv U(n)+V(n) \pmod 2, \tag{8}$$

where

$$U(n)=\sum_{\substack{d|n\\ \left(d,\frac{n}{d}\right)=1\\ d^2<n}}1 \tag{9}$$

$$V(n)=\sum_{\substack{d|n\\ \left(d,\frac{d+\frac{n}{d}}{2}\right)=1\\ d^2\leq n\\ d+\frac{n}{d}\equiv 2 \pmod 4}}1. \tag{10}$$

From (9) and (10) it follows for example that for $n>1$, $n\equiv 1 \pmod 4$ $h(-4n)$ is even. For $n\equiv 3 \pmod 4$ $h(-4n)$ is even if n is not a prime power and odd if n is a prime power. Another excellent check arises from the fact that for $n\equiv 3 \pmod 8$, $n>3$, $h(-4n)$ is divisible by 3.

A different type of check is available from the relationship

$$h(-4fm^2)=h(-4f)\,m\prod_{p|m}\left\{1-\frac{1}{p}\left(\frac{-4f}{p}\right)\right\},$$

where $f>1$, $f\equiv 1, 2 \pmod 4$ and square-free.

The computation of $g_0(n)$ was checked by computing 12 $g_0(n)$ and verifying the divisibility of 12 $g_0(n)$ by 12. The computation of $g^*(n)$ was checked by the fact that always, $g^*(n)\leq \frac{1}{2}g(n)$. In addition, R. Fricke gives in [2] the first 50 or so values of $g_0(n)$, $g^*(n)$ and these agree with the values given here.

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
1	1	1	1	1	1	0	0
2	0	1	3	2	1	0	0
3	1	0	4	2	1	0	0
4	0	0	6	3	1	0	0
5	0	2	6	2	2	0	0
6	0	0	12	4	2	0	0
7	2	0	8	2	1	0	0
8	0	0	12	4	2	0	0
9	0	0	12	4	2	0	0
10	0	2	18	4	2	0	0
11	0	0	12	2	3	1	0
12	0	0	24	6	2	0	0
13	2	2	14	2	2	0	0
14	0	0	24	4	4	1	0
15	0	0	24	4	2	1	0
16	0	0	24	6	2	0	0
17	0	2	18	2	4	1	0
18	0	0	36	8	2	0	0
19	2	0	20	2	3	1	0
20	0	0	36	6	4	1	0
21	2	0	32	4	4	1	0
22	0	0	36	4	2	2	1
23	0	0	24	2	3	2	0
24	0	0	48	8	4	1	0
25	0	2	30	6	2	0	0
26	0	2	42	4	6	2	0
27	0	0	36	6	3	1	0
28	0	0	48	6	2	2	1
29	0	2	30	2	6	2	0
30	0	0	72	8	4	3	1
31	2	0	32	2	3	2	0
32	0	0	48	8	4	1	0
33	0	0	48	4	4	3	1
34	0	2	54	4	4	3	1
35	0	0	48	4	6	3	0
36	0	0	72	12	4	1	0
37	2	2	38	2	2	2	1
38	0	0	60	4	6	4	1
39	2	0	56	4	4	3	0
40	0	0	72	8	4	3	1
41	0	2	42	2	8	3	0
42	0	0	96	8	4	5	2
43	2	0	44	2	3	3	1
44	0	0	72	6	6	4	1
45	0	0	72	8	4	3	1
46	0	0	72	4	4	5	2
47	0	0	48	2	5	4	0
48	0	0	96	12	4	3	1
49	2	0	56	8	4	1	0
50	0	2	90	12	6	2	0
51	0	0	72	4	6	5	1
52	0	0	84	6	4	5	2
53	0	2	54	2	6	4	1
54	0	0	108	12	6	4	1
55	0	0	72	4	4	5	1

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
56	0	0	96	8	8	5	1
57	2	0	80	4	4	5	2
58	0	2	90	4	2	6	3
59	0	0	60	2	9	5	0
60	0	0	144	12	4	7	3
61	2	2	62	2	6	4	1
62	0	0	96	4	8	7	2
63	0	0	96	8	4	5	1
64	0	0	96	12	4	3	1
65	0	4	84	4	8	5	1
66	0	0	144	8	8	9	3
67	2	0	68	2	3	5	2
68	0	0	108	6	8	7	2
69	0	0	96	4	8	7	2
70	0	0	144	8	4	9	4
71	0	0	72	2	7	6	0
72	0	0	144	16	4	5	2
73	2	2	74	2	4	5	2
74	0	2	114	4	10	8	2
75	0	0	120	12	6	5	1
76	0	0	120	6	6	8	3
77	0	0	96	4	8	7	2
78	0	0	168	8	4	11	5
79	2	0	80	2	5	6	1
80	0	0	144	12	8	7	2
81	0	0	108	12	6	4	1
82	0	2	126	4	4	9	4
83	0	0	84	2	9	7	1
84	0	0	192	12	8	11	4
85	0	4	108	4	4	7	3
86	0	0	132	4	10	10	3
87	0	0	120	4	6	9	2
88	0	0	144	8	4	9	4
89	0	2	90	2	12	7	1
90	0	0	216	16	8	11	4
91	4	0	112	4	6	7	2
92	0	0	144	6	6	10	4
93	2	0	128	4	4	9	4
94	0	0	144	4	8	11	4
95	0	0	120	4	8	9	1
96	0	0	192	16	8	9	3
97	2	2	98	2	4	7	3
98	0	0	168	16	8	7	2
99	0	0	144	8	6	9	3
100	0	0	180	18	4	7	3
101	0	2	102	2	14	8	1
102	0	0	216	8	4	15	7
103	2	0	104	2	5	8	2
104	0	0	168	8	12	11	3
105	0	0	192	8	8	13	5
106	0	2	162	4	6	12	5
107	0	0	108	2	9	9	2
108	0	0	216	18	6	10	4
109	2	2	110	2	6	8	3
110	0	0	216	8	12	15	5
111	2	0	152	4	8	11	2
112	0	0	192	12	4	11	5
113	0	2	114	2	8	9	3
114	0	0	240	8	8	17	7
115	0	0	144	4	6	11	4
116	0	0	180	6	12	13	4
117	0	0	168	8	8	11	4
118	0	0	180	4	6	14	6
119	0	0	144	4	10	11	1
120	0	0	288	16	8	17	7
121	0	0	132	12	6	6	2
122	0	2	186	4	10	14	5
123	0	0	168	4	6	13	5
124	0	0	192	6	6	14	6
125	0	2	150	10	10	8	2
126	0	0	288	16	8	17	7
127	2	0	128	2	5	10	3
128	0	0	192	16	8	9	3
129	2	0	176	4	12	13	4
130	0	4	252	8	4	17	8
131	0	0	132	2	15	11	1
132	0	0	288	12	8	19	8
133	4	0	160	4	4	11	5
134	0	0	204	4	14	16	5
135	0	0	216	12	6	13	4

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
136	0	0	216	8	8	15	6
137	0	2	138	2	8	11	4
138	0	0	288	8	8	21	9
139	2	0	140	2	9	11	3
140	0	0	288	12	12	19	7
141	0	0	192	4	8	15	6
142	0	0	216	4	4	17	8
143	0	0	168	4	10	13	2
144	0	0	288	24	8	13	5
145	0	4	180	4	8	13	5
146	0	2	222	4	16	17	5
147	2	0	224	16	6	11	4
148	0	0	228	6	4	17	8
149	0	2	150	2	14	12	3
150	0	0	360	24	8	19	8
151	2	0	152	2	7	12	3
152	0	0	240	8	12	17	6
153	0	0	216	8	8	15	6
154	0	0	288	8	8	21	9
155	0	0	192	4	12	15	4
156	0	0	336	12	8	23	10
157	2	2	158	2	6	12	5
158	0	0	240	4	8	19	8
159	0	0	216	4	10	17	4
160	0	0	288	16	8	17	7
161	0	0	192	4	16	15	4
162	0	0	324	24	6	16	7
163	2	0	164	2	3	13	6
164	0	0	252	6	16	19	6
165	0	0	288	8	8	21	9
166	0	0	252	4	10	20	8
167	0	0	168	2	11	14	2
168	0	0	384	16	8	25	11
169	2	2	182	14	6	8	3
170	0	4	324	8	12	23	9
171	0	0	240	8	12	17	5
172	0	0	264	6	6	20	9
173	0	2	174	2	14	14	4
174	0	0	360	8	12	27	11
175	0	0	240	12	6	15	5
176	0	0	288	12	12	19	7
177	0	0	240	4	4	19	9
178	0	2	270	4	8	21	9
179	0	0	180	2	15	15	3
180	0	0	432	24	8	25	11
181	2	2	182	2	10	14	5
182	0	0	336	8	12	25	10
183	2	0	248	4	8	19	6
184	0	0	288	8	8	21	9
185	0	4	228	4	16	17	5
186	0	0	384	8	12	29	12
187	0	0	216	4	6	17	7
188	0	0	288	6	10	22	9
189	0	0	288	12	12	19	7
190	0	0	360	8	4	27	13
191	0	0	192	2	13	16	2
192	0	0	384	24	8	21	9
193	2	2	194	2	4	15	7
194	0	2	294	4	20	23	7
195	0	0	336	8	12	25	9
196	0	0	336	24	8	17	7
197	0	2	198	2	10	16	6
198	0	0	432	16	8	29	13
199	2	0	200	2	9	16	4
200	0	0	360	24	12	19	7
201	2	0	272	4	12	21	8
202	0	2	306	4	6	24	11
203	0	0	240	4	12	19	6
204	0	0	432	12	12	31	13
205	0	4	252	4	8	19	8
206	0	0	312	4	20	25	8
207	0	0	288	8	6	21	8
208	0	0	336	12	8	23	10
209	0	0	240	4	20	19	5
210	0	0	576	16	8	41	19
211	2	0	212	2	9	17	6
212	0	0	324	6	12	25	10
213	0	0	288	4	8	23	10
214	0	0	324	4	6	26	12
215	0	0	264	4	14	21	4

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
216	0	0	432	24	12	25	10
217	4	0	256	4	8	19	8
218	0	2	330	4	10	26	11
219	2	0	296	4	12	23	8
220	0	0	432	12	8	31	14
221	0	4	252	4	16	19	6
222	0	0	456	8	12	35	15
223	2	0	224	2	7	18	6
224	0	0	384	16	16	25	9
225	0	0	360	24	8	19	8
226	0	2	342	4	8	27	12
227	0	0	228	2	15	19	5
228	0	0	480	12	8	35	16
229	2	2	230	2	10	18	7
230	0	0	432	8	20	33	12
231	0	0	384	8	12	29	9
232	0	0	360	8	4	27	13
233	0	2	234	2	12	19	7
234	0	0	504	16	12	35	15
235	0	0	288	4	6	23	10
236	0	0	360	6	18	28	10
237	2	0	320	4	12	25	10
238	0	0	432	8	8	33	15
239	0	0	240	2	15	20	3
240	0	0	576	24	8	37	17
241	2	2	242	2	12	19	7
242	0	0	396	24	10	22	9
243	0	0	324	18	9	19	7
244	0	0	372	6	12	29	12
245	0	0	336	16	12	21	8
246	0	0	504	8	12	39	17
247	4	0	280	4	6	21	8
248	0	0	384	8	16	29	11
249	0	0	336	4	12	27	11
250	0	2	450	20	10	28	12
251	0	0	252	2	21	21	4
252	0	0	576	24	8	37	17
253	0	0	288	4	4	23	11
254	0	0	384	4	16	31	12
255	0	0	432	8	12	33	11
256	0	0	384	24	8	21	9
257	0	2	258	2	16	21	7
258	0	0	528	8	8	41	19
259	4	0	304	4	12	23	8
260	0	0	504	12	16	37	15
261	0	0	360	8	12	27	11
262	0	0	396	4	6	32	15
263	0	0	264	2	13	22	5
264	0	0	576	16	16	41	17
265	0	4	324	4	8	25	11
266	0	0	480	8	20	37	14
267	0	0	360	4	6	29	13
268	0	0	408	6	6	32	15
269	0	2	270	2	22	22	6
270	0	0	648	24	12	43	19
271	2	0	272	2	11	22	6
272	0	0	432	12	16	31	12
273	4	0	448	8	8	33	15
274	0	2	414	4	12	33	14
275	0	0	360	12	12	25	9
276	0	0	576	12	16	43	18
277	2	2	278	2	6	22	10
278	0	0	420	4	14	34	14
279	0	0	384	8	12	29	9
280	0	0	576	16	8	41	19
281	0	2	282	2	20	23	7
282	0	0	576	8	8	45	21
283	2	0	284	2	9	23	9
284	0	0	432	6	14	34	14
285	0	0	480	8	16	37	15
286	0	0	504	8	12	39	17
287	0	0	336	4	14	27	7
288	0	0	576	32	8	33	15
289	0	2	306	18	8	17	7
290	0	4	540	8	20	41	16
291	2	0	392	4	12	31	12
292	0	0	444	6	8	35	16
293	0	2	294	2	18	24	8
294	0	0	672	32	12	41	18
295	0	0	360	4	8	29	11
296	0	0	456	8	20	35	13
297	0	0	432	12	12	31	13
298	0	2	450	4	6	36	17
299	0	0	336	4	24	27	6
300	0	0	720	36	12	43	19

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
301	4	0	352	4	8	27	12
302	0	0	456	4	12	37	16
303	0	0	408	4	10	33	12
304	0	0	480	12	12	35	15
305	0	4	372	4	16	29	11
306	0	0	648	16	16	47	20
307	2	0	308	2	9	25	10
308	0	0	576	12	16	43	18
309	2	0	416	4	12	33	14
310	0	0	576	8	8	45	21
311	0	0	312	2	19	26	4
312	0	0	672	16	8	49	23
313	2	2	314	2	8	25	11
314	0	2	474	4	26	38	13
315	0	0	576	16	12	41	17
316	0	0	480	6	10	38	17
317	0	2	318	2	10	26	11
318	0	0	648	8	12	51	23
319	0	0	360	4	10	29	10
320	0	0	576	24	16	37	15
321	0	0	432	4	20	35	13
322	0	0	576	8	8	45	21
323	0	0	360	4	12	29	11
324	0	0	648	36	12	37	16
325	0	4	420	12	12	29	12
326	0	0	492	4	22	40	15
327	2	0	440	4	12	35	12
328	0	0	504	8	8	39	18
329	0	0	384	4	24	31	10
330	0	0	864	16	8	65	31
331	2	0	332	2	9	27	11
332	0	0	504	6	18	40	16
333	0	0	456	8	8	35	16
334	0	0	504	4	12	41	18
335	0	0	408	4	18	33	8
336	0	0	768	24	16	53	23
337	2	2	338	2	8	27	12
338	0	2	546	28	14	32	13
339	0	0	456	4	18	37	13
340	0	0	648	12	8	49	23
341	0	0	384	4	28	31	9
342	0	0	720	16	12	53	24
343	2	0	392	14	7	26	10
344	0	0	528	8	20	41	16
345	0	0	576	8	8	45	21
346	0	2	522	4	10	42	19
347	0	0	348	2	15	29	10
348	0	0	720	12	12	55	25
349	2	2	350	2	14	28	11
350	0	0	720	24	16	49	21
351	0	0	504	12	12	37	13
352	0	0	576	16	8	41	19
353	0	2	354	2	16	29	11
354	0	0	720	8	16	57	25
355	0	0	432	4	12	35	14
356	0	0	540	6	24	43	16
357	0	0	576	8	8	45	21
358	0	0	540	4	6	44	21
359	0	0	360	2	19	30	6
360	0	0	864	32	16	57	25
361	2	0	380	20	10	22	9
362	0	2	546	4	18	44	18
363	0	0	528	24	12	33	13
364	0	0	672	12	12	51	23
365	0	4	444	4	20	35	13
366	0	0	744	8	12	59	27
367	2	0	368	2	9	30	11
368	0	0	576	12	12	43	19
369	0	0	504	8	16	39	16
370	0	4	684	8	12	53	24
371	0	0	432	4	24	35	10
372	0	0	768	12	8	59	28
373	2	2	374	2	10	30	13
374	0	0	648	8	28	51	19
375	0	0	600	20	10	41	16
376	0	0	576	8	16	45	19
377	0	4	420	4	16	33	13
378	0	0	864	24	12	61	28
379	2	0	380	2	9	31	13
380	0	0	720	12	16	55	24
381	2	0	512	4	20	41	16
382	0	0	576	4	8	47	22
383	0	0	384	2	17	32	8
384	0	0	768	32	16	49	21
385	0	0	576	8	8	45	21

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$g^*(n)$
386	0	2	582	4	20	47	19
387	0	0	528	8	12	41	17
388	0	0	588	6	8	47	22
389	0	2	390	2	22	32	11
390	0	0	1008	16	16	77	35
391	0	0	432	4	14	35	11
392	0	0	672	32	16	41	17
393	0	0	528	4	12	43	19
394	0	2	594	4	10	48	22
395	0	0	480	4	24	39	12
396	0	0	864	24	12	61	28
397	2	2	398	2	6	32	15
398	0	0	600	4	20	49	20
399	4	0	640	8	16	49	17
400	0	0	720	36	8	43	20
401	0	2	402	2	20	33	12
402	0	0	816	8	16	65	29
403	4	0	448	4	6	35	16
404	0	0	612	6	28	49	18
405	0	0	648	24	12	43	19
406	0	0	720	8	16	57	25
407	0	0	456	4	16	37	11
408	0	0	864	16	8	65	31
409	2	2	410	2	16	33	13
410	0	4	756	8	16	59	26
411	0	0	552	4	18	45	17
412	0	0	624	6	10	50	23
413	0	0	480	4	20	39	15
414	0	0	864	16	16	65	29
415	0	0	504	4	10	41	16
416	0	0	672	16	24	49	19
417	2	0	560	4	12	45	20
418	0	0	720	8	8	57	27
419	0	0	420	2	27	35	9
420	0	0	1152	24	16	85	39
421	2	2	422	2	10	34	15
422	0	0	636	4	10	52	24
423	0	0	576	8	10	45	18
424	0	0	648	8	12	51	23
425	0	4	540	12	24	39	14
426	0	0	864	8	24	69	29
427	4	0	496	4	6	39	18
428	0	0	648	6	18	52	22
429	0	0	672	8	16	53	23
430	0	0	792	8	12	63	29
431	0	0	432	2	21	36	8
432	0	0	864	36	12	55	25
433	2	2	434	2	12	35	15
434	0	0	768	8	24	61	25
435	0	0	720	8	12	57	25
436	0	0	660	6	12	53	24
437	0	0	480	4	20	39	15
438	0	0	888	8	8	71	34
439	2	0	440	2	15	36	11
440	0	0	864	16	24	65	27
441	0	0	672	32	16	41	17
442	0	4	756	8	8	59	28
443	0	0	444	2	15	37	14
444	0	0	912	12	16	71	32
445	0	4	540	4	8	43	20
446	0	0	672	4	32	55	20
447	0	0	600	4	14	49	18
448	0	0	768	24	8	53	25
449	0	2	450	2	20	37	14
450	0	0	1080	48	12	67	31
451	0	0	504	4	18	41	15
452	0	0	684	6	16	55	24
453	2	0	608	4	12	49	22
454	0	0	684	4	14	56	25
455	0	0	672	8	20	53	17
456	0	0	960	16	16	73	33
457	2	2	458	2	8	37	17
458	0	2	690	4	26	56	22
459	0	0	648	12	18	49	19
460	0	0	864	12	12	67	31
461	0	2	462	2	30	38	12
462	0	0	1152	16	8	89	43
463	2	0	464	2	7	38	16
464	0	0	720	12	24	55	22
465	0	0	768	8	16	61	27
466	0	2	702	4	8	57	27
467	0	0	468	2	21	39	13
468	0	0	1008	24	16	73	33
469	4	0	544	4	16	43	18
470	0	0	864	8	20	69	30

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$g^*(n)$
471	2	0	632	4	16	51	18
472	0	0	720	8	12	57	26
473	0	0	528	4	12	43	19
474	0	0	960	8	20	77	34
475	0	0	600	12	12	45	19
476	0	0	864	12	20	67	29
477	0	0	648	8	12	51	23
478	0	0	720	4	8	59	28
479	0	0	480	2	25	40	8
480	0	0	1152	32	16	81	37
481	4	4	532	4	16	41	17
482	0	2	726	4	20	59	25
483	0	0	768	8	12	61	27
484	0	0	792	36	12	49	22
485	0	4	588	4	20	47	19
486	0	0	972	36	18	64	28
487	2	0	488	2	7	40	17
488	0	0	744	8	20	59	25
489	2	0	656	4	20	53	22
490	0	0	1008	32	12	69	32
491	0	0	492	2	27	41	12
492	0	0	1008	12	12	79	37
493	0	4	540	4	12	43	19
494	0	0	840	8	28	67	27
495	0	0	864	16	16	65	25
496	0	0	768	12	12	59	27
497	0	0	576	4	24	47	18
498	0	0	1008	8	8	81	39
499	2	0	500	2	9	41	18
500	0	0	900	30	20	61	26
501	0	0	672	4	16	55	24
502	0	0	756	4	14	62	28
503	0	0	504	2	21	42	11
504	0	0	1152	32	16	81	37
505	0	4	612	4	8	49	23
506	0	0	864	8	28	69	28
507	2	0	728	28	12	47	20
508	0	0	768	6	10	62	29
509	0	2	510	2	30	42	14
510	0	0	1296	16	16	101	47
511	4	0	592	4	14	47	17
512	0	0	768	32	16	49	21
513	0	0	720	12	12	55	25
514	0	2	774	4	16	63	28
515	0	0	624	4	18	51	20
516	0	0	1056	12	24	83	36
517	0	0	576	4	12	47	21
518	0	0	912	8	16	73	33
519	0	0	696	4	18	57	20
520	0	0	1008	16	8	77	37
521	0	2	522	2	32	43	14
522	0	0	1080	16	8	83	40
523	2	0	524	2	15	43	17
524	0	0	792	6	30	64	25
525	0	0	960	24	16	69	31
526	0	0	792	4	12	65	30
527	0	0	576	4	18	47	15
528	0	0	1152	24	16	85	39
529	0	0	552	24	12	35	15
530	0	4	972	8	28	77	32
531	0	0	720	8	18	57	23
532	0	0	960	12	8	75	36
533	0	4	588	4	12	47	21
534	0	0	1080	8	20	87	39
535	0	0	648	4	14	53	20
536	0	0	816	8	28	65	26
537	0	0	720	4	12	59	27
538	0	2	810	4	10	66	31
539	0	0	672	16	24	49	17
540	0	0	1296	36	12	91	43
541	2	2	542	2	10	44	20
542	0	0	816	4	24	67	28
543	2	0	728	4	12	59	24
544	0	0	864	16	16	65	29
545	0	4	660	4	32	53	19
546	0	0	1344	16	24	105	47
547	2	0	548	2	9	45	20
548	0	0	828	6	16	67	30
549	0	0	744	8	24	59	24
550	0	0	1080	24	12	79	37
551	0	0	600	4	26	49	12
552	0	0	1152	16	16	89	41
553	4	0	640	4	8	51	24
554	0	2	834	4	22	68	29
555	0	0	912	8	12	73	33

667805—63——5

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$g^*(n)$
556	0	0	840	6	18	68	30
557	0	2	558	2	18	46	19
558	0	0	1152	16	16	89	41
559	4	0	616	4	16	49	17
560	0	0	1152	24	24	85	37
561	0	0	864	8	16	69	31
562	0	2	846	4	8	69	33
563	0	0	564	2	27	47	15
564	0	0	1152	12	16	91	42
565	0	4	684	4	12	55	25
566	0	0	852	4	30	70	28
567	0	0	864	24	12	61	25
568	0	0	864	8	8	69	33
569	0	2	570	2	32	47	16
570	0	0	1440	16	16	113	53
571	2	0	572	2	15	47	19
572	0	0	1008	12	20	79	35
573	0	0	768	4	16	63	28
574	0	0	1008	8	16	81	37
575	0	0	720	12	18	55	19
576	0	0	1152	48	16	73	23
577	2	2	578	2	8	47	22
578	0	2	918	36	16	59	26
579	2	0	776	4	24	63	24
580	0	0	1080	12	16	85	39
581	0	0	672	4	28	55	21
582	0	0	1176	8	16	95	44
583	0	0	648	4	8	53	23
584	0	0	888	8	32	71	28
585	0	0	1008	16	16	77	35
586	0	2	882	4	18	72	22
587	0	0	588	2	21	49	18
588	0	0	1344	48	12	89	42
589	4	0	610	4	16	51	22
590	0	0	1080	8	20	87	39
591	0	0	792	4	22	65	22
592	0	0	912	12	8	71	34
593	0	2	594	2	24	49	19
594	0	0	1296	24	24	97	43
595	0	0	864	8	12	69	31
596	0	0	900	6	28	73	30
597	2	0	800	4	12	65	30
598	0	0	1008	8	8	81	39
599	0	0	600	2	25	50	13
600	0	0	1440	48	16	97	45
601	2	2	602	2	20	49	20
602	0	0	1056	8	24	85	37
603	0	0	816	8	12	65	29
604	0	0	912	6	14	74	34
605	0	0	792	24	24	55	22
606	0	0	1224	8	12	99	47
607	2	0	608	2	13	50	19
608	0	0	960	16	24	73	31
609	0	0	960	8	16	77	35
610	0	4	1116	8	12	89	42
611	0	0	672	4	30	55	18
612	0	0	1296	24	16	97	45
613	2	2	614	2	10	50	23
614	0	0	924	4	34	76	30
615	0	0	1008	8	20	81	31
616	0	0	1152	16	16	89	41
617	0	2	618	2	12	51	23
618	0	0	1248	8	12	101	48
619	2	0	620	2	15	51	21
620	0	0	1152	12	24	91	40
621	0	0	864	12	24	67	28
622	0	0	936	4	12	77	36
623	0	0	720	4	22	59	19
624	0	0	1344	24	16	101	47
625	0	2	750	30	10	48	22
626	0	2	942	4	36	77	30
627	0	0	960	8	12	77	35
628	0	0	948	6	12	77	36
629	0	4	684	4	36	55	19
630	0	0	1728	32	16	129	61
631	2	0	632	2	13	52	20
632	0	0	960	8	16	77	35
633	2	0	848	4	20	69	30
634	0	2	954	4	14	78	36
635	0	0	768	4	30	63	22
636	0	0	1296	12	20	103	47
637	4	0	784	16	12	57	26
638	0	0	1080	8	20	87	39
639	0	0	864	8	14	69	28
640	0	0	1152	32	16	81	37

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$g^*(n)$
641	0	2	642	2	28	53	20
642	0	0	1296	8	16	105	49
643	2	0	644	2	9	53	24
644	0	0	1152	12	32	91	38
645	0	0	1056	8	16	85	39
646	0	0	1080	8	16	87	40
647	0	0	648	2	23	54	16
648	0	0	1296	48	12	85	40
649	0	0	720	4	20	59	25
650	0	4	1260	24	24	93	41
651	4	0	1024	8	24	81	33
652	0	0	984	6	6	80	39
653	0	2	654	2	14	54	24
654	0	0	1320	8	28	107	47
655	0	0	792	4	12	65	27
656	0	0	1008	12	32	79	32
657	0	0	888	8	16	71	32
658	0	0	1152	8	8	93	45
659	0	0	660	2	33	55	17
660	0	0	1728	24	16	133	63
661	2	2	662	2	18	54	23
662	0	0	996	4	22	82	36
663	0	0	1008	8	16	81	33
664	0	0	1008	8	20	81	36
665	0	0	960	8	24	77	33
666	0	0	1368	16	20	107	49
667	0	0	720	4	12	59	26
668	0	0	1008	6	22	82	36
669	2	0	896	4	12	73	34
670	0	0	1224	8	12	99	47
671	0	0	744	4	30	61	16
672	0	0	1536	32	16	113	53
673	2	2	674	2	12	55	25
674	0	2	1014	4	24	83	36
675	0	0	1080	36	18	73	31
676	0	0	1092	42	12	71	33
677	0	2	678	2	30	56	21
678	0	0	1368	8	20	111	51
679	4	0	784	4	18	63	23
680	0	0	1296	16	24	101	45
681	0	0	912	4	20	75	33
682	0	0	1152	8	12	93	44
683	0	0	684	2	15	57	24
684	0	0	1440	24	24	109	49
685	0	4	828	4	12	67	31
686	0	0	1176	28	28	85	36
687	2	0	920	4	12	75	32
688	0	0	1056	12	12	83	39
689	0	4	756	4	40	61	21
690	0	0	1728	16	16	137	65
691	2	0	692	2	15	57	24
692	0	0	1044	6	28	85	36
693	0	0	1152	16	16	89	41
694	0	0	1044	4	10	86	41
695	0	0	840	4	24	69	23
696	0	0	1440	16	24	113	51
697	0	4	756	4	8	61	29
698	0	2	1050	4	26	86	37
699	0	0	936	4	30	77	29
700	0	0	1440	36	12	103	49
701	0	2	702	2	34	58	21
702	0	0	1512	24	12	115	55
703	4	0	760	4	14	61	24
704	0	0	1152	24	24	85	37
705	0	0	1152	8	24	93	41
706	0	2	1062	4	24	87	38
707	0	0	816	4	18	67	28
708	0	0	1440	12	8	115	56
709	2	2	710	2	10	58	27
710	0	0	1296	8	32	105	45
711	0	0	960	8	20	77	29
712	0	0	1080	8	16	87	40
713	0	0	768	4	24	63	26
714	0	0	1728	16	24	137	63
715	0	0	1008	8	12	81	37
716	0	0	1080	6	30	88	37
717	0	0	960	4	16	79	36
718	0	0	1080	4	12	89	42
719	0	0	720	2	31	60	15
720	0	0	1728	48	16	121	57
721	4	0	832	4	16	67	30
722	0	0	1140	40	18	76	34
723	2	0	968	4	12	79	36
724	0	0	1092	6	20	89	40
725	0	4	900	12	24	69	29

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$q^*(n)$
726	0	0	1584	48	20	109	50
727	2	0	728	2	13	60	24
728	0	0	1344	16	24	105	47
729	0	0	972	36	18	64	28
730	0	4	1332	8	12	107	51
731	0	0	792	4	36	65	21
732	0	0	1488	12	16	119	56
733	2	2	734	2	14	60	27
734	0	0	1104	4	40	91	36
735	0	0	1344	32	16	97	41
736	0	0	1152	16	16	89	41
737	0	0	816	4	20	67	29
738	0	0	1512	16	16	119	56
739	2	0	740	2	15	61	26
740	0	0	1368	12	32	109	47
741	4	0	1120	8	24	89	39
742	0	0	1296	8	8	105	51
743	0	0	744	2	21	62	21
744	0	0	1536	16	24	121	55
745	0	4	900	4	16	73	33
746	0	2	1122	4	26	92	40
747	0	0	1008	8	18	81	35
748	0	0	1296	12	12	103	49
749	0	0	864	4	32	71	28
750	0	0	1800	40	20	131	61
751	2	0	752	2	15	62	24
752	0	0	1152	12	20	91	41
753	0	0	1008	4	12	83	39
754	0	4	1260	8	20	101	46
755	0	0	912	4	36	75	26
756	0	0	1728	36	24	127	58
757	2	2	758	2	10	62	29
758	0	0	1140	4	22	94	42
759	0	0	1152	8	24	93	35
760	0	0	1440	16	8	113	55
761	0	2	762	2	40	63	22
762	0	0	1536	8	12	125	60
763	4	0	880	4	12	71	32
764	0	0	1152	6	26	94	41
765	0	0	1296	16	16	101	47
766	0	0	1152	4	24	95	42
767	0	0	840	4	22	69	24
768	0	0	1536	48	16	105	49
769	2	2	770	2	20	63	27
770	0	0	1728	16	32	137	61
771	0	0	1032	4	18	85	37
772	0	0	1164	6	8	95	46
773	0	2	774	2	26	64	26
774	0	0	1584	16	20	125	58
775	0	0	960	12	12	75	32
776	0	0	1176	8	40	95	38
777	4	0	1216	8	16	97	45
778	0	2	1170	4	14	96	45
779	0	0	840	4	30	69	25
780	0	0	2016	24	24	157	73
781	0	0	864	4	20	71	31
782	0	0	1296	8	24	105	47
783	0	0	1080	12	18	85	34
784	0	0	1344	48	16	89	41
785	0	4	948	4	16	77	35
786	0	0	1584	8	16	129	61
787	2	0	788	2	15	65	28
788	0	0	1188	6	20	97	44
789	0	0	1056	4	32	87	36
790	0	0	1440	8	16	117	55
791	0	0	912	4	32	75	22
792	0	0	1728	32	16	129	61
793	4	4	868	4	8	69	33
794	0	2	1194	4	42	98	39
795	0	0	1296	8	12	105	49
796	0	0	1200	6	18	98	45
797	0	2	798	2	30	66	26
798	0	0	1920	16	16	153	73
799	0	0	864	4	16	71	28
800	0	0	1440	48	24	97	43
801	0	0	1080	8	24	87	38
802	0	2	1206	4	12	99	47
803	0	0	888	4	30	73	27
804	0	0	1632	12	24	131	60
805	0	0	1152	8	16	93	43
806	0	0	1344	8	28	109	48
807	0	0	1080	4	14	89	38
808	0	0	1224	8	12	99	47
809	0	2	810	2	32	67	26
810	0	0	1944	48	24	139	64

n	$\nu_1(n)$	$\nu_2(n)$	$\mu(n)$	$\sigma_0(n)$	$h(-4n)$	$q_0(n)$	$q^*(n)$
811	2	0	812	2	21	67	27
812	0	0	1440	12	24	115	52
813	2	0	1088	4	12	89	42
814	0	0	1368	8	12	111	53
815	0	0	984	4	30	81	26
816	0	0	1728	24	24	133	61
817	4	0	880	4	12	71	33
818	0	2	1230	4	28	101	44
819	0	0	1344	16	24	105	45
820	0	0	1512	12	16	121	57
821	0	2	822	2	30	68	27
822	0	0	1656	8	20	135	63
823	2	0	824	2	9	68	30
824	0	0	1248	8	40	101	41
825	0	0	1440	24	24	109	49
826	0	0	1440	8	12	117	56
827	0	0	828	2	21	69	28
828	0	0	1728	24	12	133	64
829	2	2	830	2	22	68	29
830	0	0	1512	8	20	123	57
831	2	0	1112	4	28	91	32
832	0	0	1344	24	16	101	47
833	0	0	1008	16	24	77	33
834	0	0	1680	8	16	137	65
835	0	0	1008	4	18	83	36
836	0	0	1440	12	40	115	48
837	0	0	1152	12	12	91	43
838	0	0	1260	4	14	104	49
839	0	0	840	2	33	70	19
840	0	0	2304	32	16	177	85
841	0	2	870	30	14	58	26
842	0	2	1266	4	26	104	46
843	0	0	1128	4	18	93	41
844	0	0	1272	6	18	104	48
845	0	4	1092	28	28	77	32
846	0	0	1728	16	32	137	61
847	0	0	1056	24	10	77	34
848	0	0	1296	12	24	103	46
849	2	0	1136	4	28	93	40
850	0	4	1620	24	16	123	58
851	0	0	912	4	30	75	28
852	0	0	1728	12	16	139	66
853	2	2	854	2	10	70	33
854	0	0	1488	8	44	121	50
855	0	0	1440	16	16	113	49
856	0	0	1296	8	12	105	50
857	0	2	858	2	32	71	28
858	0	0	2016	16	16	161	77
859	2	0	860	2	21	71	29
860	0	0	1584	12	28	127	57
861	0	0	1344	8	24	109	49
862	0	0	1296	4	8	107	52
863	0	0	864	2	21	72	26
864	0	0	1728	48	24	121	55
865	0	4	1044	4	16	85	39
866	0	2	1302	4	44	107	43
867	0	0	1224	36	18	85	37
868	0	0	1536	12	16	123	58
869	0	0	960	4	32	79	32
870	0	0	2160	16	16	173	83
871	4	0	952	4	22	77	28
872	0	0	1320	8	20	107	49
873	0	0	1176	8	16	95	44
874	0	0	1440	8	20	117	54
875	0	0	1200	20	30	91	36
876	0	0	1776	12	24	143	66
877	2	2	878	2	10	72	34
878	0	0	1320	4	20	109	50
879	0	0	1176	4	22	97	38
880	0	0	1728	24	16	133	63
881	0	2	882	2	40	73	27
882	0	0	2016	64	16	137	65
883	2	0	884	2	9	73	34
884	0	0	1512	12	32	121	53
885	0	0	1440	8	24	117	53
886	0	0	1332	4	18	110	51
887	0	0	888	2	29	74	23
888	0	0	1824	16	24	145	67
889	4	0	1024	4	16	83	38
890	0	4	1620	8	24	131	60
891	0	0	1296	24	18	97	43
892	0	0	1344	6	14	110	52
893	0	0	960	4	28	79	33
894	0	0	1800	8	28	147	67
895	0	0	1080	4	16	89	37

n	$\nu_1(n)$	$\nu_2(n)$	$\mu_0(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
896	0	0	1536	32	32	113	49
897	0	0	1344	8	16	109	51
898	0	2	1350	4	12	111	53
899	0	0	960	4	42	79	26
900	0	0	2160	72	16	145	69
901	0	4	972	4	24	79	34
902	0	0	1512	8	28	123	55
903	4	0	1408	8	16	113	49
904	0	0	1368	8	16	111	52
905	0	4	1092	4	24	89	39
906	0	0	1824	8	28	149	68
907	2	0	908	2	9	75	35
908	0	0	1368	6	30	112	49
909	0	0	1224	8	28	99	43
910	0	0	2016	16	16	161	77
911	0	0	912	2	31	76	23
912	0	0	1920	24	16	149	71
913	0	0	1008	4	12	83	39
914	0	2	1374	4	36	113	48
915	0	0	1488	8	24	121	53
916	0	0	1380	6	20	113	52
917	0	0	1056	4	20	87	39
918	0	0	1944	24	12	151	73
919	2	0	920	2	19	76	29
920	0	0	1728	16	40	137	59
921	2	0	1232	4	20	101	46
922	0	2	1386	4	18	114	53
923	0	0	1008	4	30	83	32
924	0	0	2304	24	24	181	85
925	0	4	1140	12	12	89	42
926	0	0	1392	4	40	115	48
927	0	0	1248	8	20	101	41
928	0	0	1440	16	8	113	55
929	0	2	930	2	36	77	30
930	0	0	2304	16	24	185	87
931	4	0	1120	16	18	85	37
932	0	0	1404	6	24	115	52
933	0	0	1248	4	16	103	48
934	0	0	1404	4	26	116	52
935	0	0	1296	8	28	105	39
936	0	0	2016	32	24	153	71
937	2	2	938	2	20	77	34
938	0	0	1632	8	16	133	63
939	2	0	1256	4	24	103	44
940	0	0	1728	12	12	139	67
941	0	2	942	2	46	78	28
942	0	0	1896	8	12	155	75
943	0	0	1008	4	16	83	34
944	0	0	1440	12	36	115	49
945	0	0	1728	24	24	133	61
946	0	0	1584	8	16	129	61
947	0	0	948	2	15	79	35
948	0	0	1920	12	24	155	72
949	4	4	1036	4	12	83	39
950	0	0	1800	24	36	139	61
951	0	0	1272	4	26	105	40
952	0	0	1728	16	16	137	65
953	0	2	954	2	32	79	32
954	0	0	1944	16	24	155	72
955	0	0	1152	4	12	95	44
956	0	0	1440	6	30	118	52
957	0	0	1440	8	16	117	55
958	0	0	1440	4	16	119	56
959	0	0	1104	4	36	91	28
960	0	0	2304	48	16	169	81

n	$\nu_1(n)$	$\nu_2(n)$	$\mu_0(n)$	$\sigma_0(n)$	$h(-4n)$	$g_0(n)$	$g^*(n)$
961	2	0	992	32	16	67	30
962	0	4	1596	8	28	129	58
963	0	0	1296	8	18	105	47
964	0	0	1452	6	24	119	54
965	0	4	1164	4	44	95	37
966	0	0	2304	16	24	185	87
967	2	0	968	2	11	80	35
968	0	0	1584	48	20	109	50
969	0	0	1440	8	24	117	53
970	0	4	1764	8	12	143	69
971	0	0	972	2	45	81	26
972	0	0	1944	54	18	136	64
973	4	0	1120	4	12	91	43
974	0	0	1464	4	36	121	52
975	0	0	1680	24	16	129	57
976	0	0	1488	12	24	119	54
977	0	2	978	2	20	81	36
978	0	0	1968	8	24	161	75
979	0	0	1080	4	24	89	37
980	0	0	2016	48	24	145	67
981	0	0	1320	8	24	107	48
982	0	0	1476	4	10	122	59
983	0	0	984	2	27	82	28
984	0	0	2016	16	24	161	75
985	0	4	1188	4	24	97	43
986	0	4	1620	8	44	131	55
987	0	0	1536	8	24	125	55
988	0	0	1680	12	12	135	65
989	0	0	1056	4	36	87	35
990	0	0	2592	32	24	201	95
991	2	0	992	2	17	82	33
992	0	0	1536	16	32	121	53
993	2	0	1328	4	12	109	52
994	0	0	1728	8	16	141	67
995	0	0	1200	4	24	99	42
996	0	0	2016	12	24	163	76
997	2	2	998	2	14	82	38
998	0	0	1500	4	26	124	56
999	0	0	1368	12	24	109	43
1000	0	0	1800	40	20	131	61

References

[1] L. R. Ford, Automorphic functions, Second edition, New York (1951).

[2] R. Fricke, Lehrbuch der Algebra, Dritter Band, Braunschweig (1928).

[3] R. Gunning, Lectures on modular forms, Princeton (1962).

[4] E. Hecke, Analytische Arithmetik der positiven quadratischen Formen, Kgl. Danske Videnskabernes Selskab. Mathematiskfysiske Meddelelser **17,** 12 (1940).

[5] E. Hecke, Die Klassenzahl imaginar—quadratischer Körper in der Theorie der elliptischen Modulfunktionen. Monatshefte für Mathematik und Physik **48,** 75–83 (1959).

[6] F. Klein, Gesammelte mathematische Abhandlungen, Dritter Band, Berlin (1923).

(Paper 67B1–93)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 1, January–March 1963

Publications of the National Bureau of Standards*

Selected Abstracts

Tables of Einstein functions, Vibrational contributions to the thermodynamic functions, J. Hilsenrath and G. G. Ziegler, *NBS Mono. 49, (July 12, 1962) $2.75.*
Tables are presented for the contribution of a harmonic oscillator to the free energy function, enthalpy function, entropy, and heat capacity of gases. Dimensionless values of the Planck-Einstein functions are given as a function of $x=hc\nu/kT$ for $x=0.0010(.0001)$ °K directly as a function of frequency ν, and temperature T, was computed using the values 1.43880 for the second radiation constant hc/k, and 1.98717 for the universal gas constant R. The arguments for the latter table are spaced at 10 wave number intervals from 100 cm^{-1} to 400 cm^{-1}.

Systems of electrical units, F. B. Silsbee, *NBS Mono. 56, (Sept. 20, 1962) 30 cents.*
The various systems of measurement, with their respective sets of units, used in the literature on electricity and magnetism are described in detail. Their historical development is summarized. The manner in which each is derived from either of the two alternative points of view of the experimentalist and the theoretician is compared and contrasted. The desirability of recognizing both points of view in international standardization, particularly when discussing rationalization, is pointed out. The present status of the absolute measurements on which all electrical units are based is reported, and tables are included for the conversion of equations and numerical values from one system to another.

Tchebycheff approximations by exponentials, J. R. Rice, *J. Soc. Ind. Appl. Math.* **10,** *No. 1, 149–161 (Mar. 1962).*
Tchebycheff approximations of the form

$$\sum_{i=1}^{k}\sum_{j=0}^{m_i} p_{ij}x^j b_i^x$$

where $\sum_{i=1}^{k}(m_i+1)\leq n$ are studied. With this approximating function it is shown that a best approximation on [0,1] exists for every continuous function. It is shown that this function is a varisolvent function and the uniqueness and characteristic properties of best approximations follow from known results for varisolvent functions. A theorem relating the best approximation on [0,1] is established for varisolvent functions, and hence for this approximating function. The generalization to an approximating function which is an arbitrary linear combination of exponentials with polynomial coefficients is given.

Real representations of coordinate rotations, U. Fano, *J. Math. Phys.* **1,** *No. 5, 417–423 (Sept.–Oct. 1960).*
Since irreducible tensorial sets that represent observables are of integral degree, their transformations under coordinate rotations have real representations. Real representations, with rows and columns classified by eigenvalues of the commuting operators J_z^2 and $\exp(i\tau J_z)$, are given explicitly, so that complex functions of rotation angles need not be used. The addition of angular momenta is worked out for sets in the real representation such as the sets of real orbital wave functions. Applications to the theory of angular distributions are discussed.

The vapor pressure of 20 °K equilibrium hydrogen, L. C. Weber, Jr., D. E. Diller, H. M. Roder, and R. D. Goodwin, *cryogenics* **3,** *236–238 (June 1962).*
The vapor pressure of 20 °K equilibrium hydrogen has been measured between the normal boiling point and the critical point. The data are represented analytically for purposes of smoothing and interpolation and the results are compared with earlier work.

Studies in non-equilibrium rate processes. V. The relaxation of moments derived from a master equation, K. E. Shuler, K. Anderson, and G. H. Weiss, *J. Math. Phys.* **3,** *No. 3, 550–556 (May–June 1962).*
A study has been made of the relaxation of the moments of probability distributions whose time evolution are governed by a master equation. The necessary and sufficient conditions for the first moment to undergo a simple exponential relaxation have been derived. The necessary and sufficient conditions under which the first k moments satisfy a closed system of linear equations also have been developed. This treatment has also been extended to an investigation of the moments of the distribution determined by the Fokker-Planck equation.

Irreversible processes in liquids and the density matrix: monatomic molecules, R. E. Nettleton, *Phys. Fluids* **5,** *No. 6, 687–700 (June 1962).*
A small subvolume of an infinite, isotropic, monatomic liquid is approximated by a model in which no diffusion is permitted across the boundary, and interaction with the surroundings is represented by a random force potential in the Hamiltonian. The density matrix for this model is expanded in a sum of scalar, vector, and tensor parameters, for which rate equations are derived from the quantum Liouville equation. An interpretation of these rate equations as phenomenological equations in the sense of irreversible thermodynamics leads to a set of conditions which can be used to determine the molecular configuration dependence of the density matrix and the stochastic interaction term in the Hamiltonian. The result provides a generalization, based on quantum statistics, of a previous, purely phenomenological, extension of irreversible thermodynamics to include inertial effects.

Graphs for bivariate normal probabilities, M. Zelen and N. C. Severo, *Ann. Math. Stat.* **31,** *No. 3, 619–624 (Sept. 1960).*
A chart is presented which enables one to quickly calculate the bivariate normal probability integral

$$L(h,k;\rho)=\int_h^\infty dx\int_k^\infty g(x,y;\rho)dy$$

$$\text{where } g(x,y;\rho)=[2\Pi\sqrt{1-\rho^2}]^{-1}\exp\ -\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)$$

The use of this chart will result in at most an error of 10^{-2}. Examples of application are included.

Oscillatory phenomena, A. H. Kahn (*Proc. Conf. High Magnetic Field, Mass. Inst. Tech., Cambridge, Mass., Nov. 1–4, 1961), Book, High Magnetic Fields, pp. 480–488 (Mass. Inst. Tech. Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, N.Y., 1962).*
A quantum treatment of electrical conductivity in high magnetic fields will be presented. It will be shown how the transport coefficients and the magnetic susceptibility display variations periodic in the reciprocal of the magnetic field strength. The use of this high field behavior in the determination of electronic band structures will be discussed. The scattering of conduction electrons by impurities is greatly modified by the magnetic field. The effect on conductivity will be discussed.

Games associated with a renewal process, M. M. Siddiqui, *Ann. Math. Stat.* **31,** *697–701 (June 1962).*
Robbins (Annals of Mathematical Statistics, Vol. 32 (1961),

pp. 187–194) considered games associated with a recurrent event, when the trials are being performed at discrete times so that the interval between two events is an integer. His results are extended to the situations where the interval between two events is not necessarily an integer, such as the interval between arrivals of two meteors within the atmosphere.

Criteria for the reality of matrix eigenvalues, M. P. Drazin and E. V. Haynesworth, *Z. Math.* **78,** *449–452 (Mar. 1962).*
This note is concerned with generalizations of the known fact that an $n \times n$ matrix with n linearly independent eigenvectors all corresponding to real eigenvalues is similar to a hermitian matrix, and can consequently be transformed into its conjugate transpose by a hermitian similarity. For any positive integer m, an analogous necessary and sufficient is established for a given square complex matrix A to have a set of real eigenvalues (not necessarily all distinct) to which there correspond at least m linearly independent eigenvectors; analogous results about eigenvalues of modulus unity and pure imaginary eigenvalues are also obtained.
The above-mentioned results are applied to establish a theorem on the reality of the eigenvalues of a certain type of matrix, and to give a new proof of a theorem of K. Goldberg; but in fact Goldberg's defining conditions for the class of matrices in question are very dissimilar from ours, so the point of interest is rather the (essential) equivalence of two definitions. Another application leads to theorems about the reality of the roots of an arbitrary complex polynomial $h(\lambda)$, which refine and extend previously known theorems along similar lines.

Group theory and crystal field theory, C. M. Herzfeld and P. H. E. Meijer, *Book, Solid State Physics* **12,** *1–91 (Academic Press, Inc., New York, N.Y., 1961).*
A survey of the principal group theoretical principles and methods used in crystal field theory.

Algorithm and rapid binary division, R. S. Ledley and J. B. Wilson, *IRE Trans. Electronic Computers* **EC–10,** *No. 4, 662–670 (Dec. 1961).*
An algorithm for performing rapid division with binary numbers is described. First a partial algorithm is presented that has the advantage of extreme simplicity, and in a majority of the cases significantly reduces the number of operations required for division. Second, the complete algorithm is given that produces the minimum number of operations for division, which will result on the average in a saving of nearly two thirds the normal division time in digital computers.

A procedure for estimating eigenvalues, N. W. Bazley and D. Fox, *J. Math. Phys.* **3,** *469–471 (May–June 1962).*
A new procedure is given for calculation of lower bounds to the eigenvalues of self-adjoint operators. Calculation of the lower bounds is reduced to the solution of linear algebraic problems.

Structural and internal state variables in the description of scalar rate processes in fluids, R. E. Nettleton, *Phys. Fluids* **4,** *1488 (1961).*
Arguments are presented to show that the internal energy per molecule in a macroscopically small volume element of an infinite liquid should remain constant, in first approximation, during a sudden fluctuation in liquid structure at constant density and temperature. This result is shown to be consistent with a formulation of non-equilibrium thermodynamics in which the departures of structural parameters from their local equilibrium values appear as thermodynamic variables and in which there is no relaxing structural specific heat. However, it is shown that such a relaxing specific heat must appear in the thermodynamic treatment of thermal variables, which give the populations of internal vibrational-rotational states; and a new non-equilibrium thermodynamics of thermal relaxation, extending the earlier work of Meixner to include inertial effects, is formulated in such a way as to be consistent with this fact. The new formulation, which replaces a previous theory now shown to be valid only for structural variables, is based on the Pauli equation, generalized to include second-order time-derivatives. It is shown, on plausible assumptions about the molecular transition probabilities, that one can calculate all the rate constants and relaxation times introduced to describe inertial effects, as well as the thermodynamic forces.

Quantum-mechanical calculation of harmonic oscillator transition probabilities in a one-dimensional impulsive collision, K. E. Shuler and R. W. Zwanzig, *J. Chem. Phys.* **33,** *No. 6, 1778–1784 (Dec. 1960).*
The quantum mechanical vibrational transition probabilities $P_{i \to f}(\epsilon)$ for harmonic oscillators, undergoing impulsive hard sphere collisions along the line of centers with an incident beam of atoms with relative kinetic energy ϵ, have been computed by a machine (IBM–704) solution of the relevant Schrödinger equation. Curves for $P_{i \to f}(\epsilon)$ over a range of ϵ are presented for initial (i) and final (f) vibrational oscillator states $i, f = 0, 1, 2,$ and 3. It is shown that this model of an inelastic collision gives rise to appreciable vibrational transitions $\nu(i) \to \nu(f)$ with $|\Delta\nu| > 1$ (in addition to the $|\Delta\nu| = 1$ transitions) in contrast to the Landau-Teller-Herzfeld adiabatic, 1st order perturbation treatment which permits only transitions for which $|\Delta\nu| = 1$. The implication of this result is discussed in relation to the dissociation of diatomic molecules and to the adsorption of atoms on solids. Averaged transition probabilities $\overline{P}_{i \to f}(T)$ are computed for an incident beam of particles with a Maxwellian velocity distribution. It is pointed out that such averaged transition probabilities may give a misleading impression of the efficiency of translational-vibrational energy transfer if the $P_{i \to f}(\epsilon)$ show a resonance type of behavior, i.e., a strong order of magnitude dependence of $P_{i \to f}(\epsilon)$ on ϵ over a small interval of ϵ.

Other NBS Publications

J. Research NBS 66A (Phys. and Chem.), No. 6 (Nov.–Dec. 1962), 70 cents.
Heat of formation of nitronium perchlorate. A. A. Gilliland.
Phase equilibrium relations in the binary system bismuth sesquioxide-niobium pentoxide. R. S. Roth and J. L. Waring.
Elastic constants of rutile (TiO_2). J. B. Wachtman, Jr., W. E. Tefft, and D. G. Lam, Jr.
Reaction of hardened portland cement paste with carbon dioxide. C. M. Hunt and L. A. Tomes.
Titanium standards for hydrogen content. J. T. Sterling, F. J. Palumbo, and L. L. Wyman.
Ultraviolet stability of crosslinked polycaprolactam systems. S. D. Bruck.
Spectral-line intensities and gf-values in the first spectrum of copper. C. H. Corliss.
Batch adsorption from solution. W. V. Loebenstein.
Separation of hafnium from zirconium and their determination: Separation by anion-exchange. L. A. Machlan and J. L. Hague.

J. Research NBS 67A (Phys. and Chem.), No. 1 (Jan.–Feb. 1963). 70 cents.
Heat of formation of calcium aluminate monosulfate at 25 °C. H. A. Berman and E. S. Newman.
2,3-Dimethylpentane and 2-methylhexane as a test mixture for evaluating highly efficient fractionating columns. E. C. Kuehner.
Phase equilibrium relations in the Sc_2O_3-Ga_2O_3 system. S. J. Schneider and J. L. Waring.
Analysis of two infrared bands of CH_2D_2. W. B. Olson, H. C. Allen, Jr., and E. K. Plyler.
Precise coulometric titrations of halides. G. Marinenko and J. K. Taylor.
Radial distribution study of vitreous barium borosilicate. G. J. Piermarini and S. Block.
Dynamic compressibility of poly(vinyl acetate) and its relation to free volume. J. E. McKinney and H. V. Belcher.
An investigation of the constitution of the mercury-tin system. D. F. Taylor and C. L. Burns.
Effect of methyl bromide additions on the flame speed of methane. C. Halpern.

J. Research NBS 67C (Eng. and Instr.), No. 1 (Jan.–Mar. 1963), 75 cents.

Method for calibrating a standard volt box. B. L. Dunfee.

Stability of residual thiosulfate in processed microfilm. C. I. Pope.

Equipment for single crystal growth from aqueous solution. J. L. Torgesen, A. T. Horton, and C. P. Saylor.

An automatic multichannel correlator. R. F. Brown, Jr.

Elastic constant—porosity relations for polycrystalline thoria. S. Spinner, F. P. Knudsen, and L. Stone.

An oxygen partial pressure warning instrument. L. Greenspan.

New fast-opening, large-aperture shutter for high-speed photography. E. C. Cassidy and D. H. Tsai.

Equations for the radiofrequency magnetic permeameter. C. A. Hoer and A. L. Rasmussen.

J. Research NBS 67D (Radio Prop.), No. 1 (Jan.–Feb. 1963), 70 cents.

A lunar theory reasserted—a rebuttal. J. V. Evans.

Point-to-point communication on the moon. L. E. Vogler.

HF communication during ionospheric storms. G. E. Hill.

Use of surface refractivity in the empirical prediction of total atmospheric refraction. W. R. Iliff and J. M. Holt.

Effective sunspot numbers. W. B. Chadwick.

On the theory of radio wave propagation over inhomogeneous earth. K. Furutsu.

Fields of electric dipoles in sea water (a correction). W. Anderson.

Composition of reflection and transmission formulae. J. Heading.

Titheridge coefficients for the polynomial method of deducing electron density profiles from ionograms. A. R. Long and J. O. Thomas.

Input admittance of linear antennas driven from a coaxial line. T. T. Wu.

Bibliography on atomic transition probabilities, B. M. Glennon and W. L. Wiese, NBS Mono. 50 (Aug. 1, 1962), 35 cents.

Experimental transition probabilities for spectral lines of seventy elements, C. H. Corliss and W. R. Bozman, NBS Mono. 53 (July 20, 1962), $4.25.

Chemistry of cement. Proceedings of the fourth international symposium, Washington, 1960, NBS Mono. 43, Vol. I (Aug. 31, 1962), $5.75; Vol. II (Sept. 27, 1962), $5.50. The two volumes are available at $11.25 a set.

Quarterly radio noise data, March, April, May 1962 and corrigendum for Technical Notes 18–1 through 18–11, W. Q. Crichlow, R. T. Disney, and M. A. Jenkins, NBS Tech. Note 18–14, (August 9, 1962) 50 cents.

Mean electron density variations of the quiet ionosphere, No. 8—October 1959, J. W. Wright, L. R. Wescott and D. J. Brown, NBS Tech. Note 40–8, (September 1962) 35 cents.

Synoptic radio metrology, B. R. Bean, J. D. Horn, and L. P. Riggs, NBS Tech. Note 98, (October 1962) 50 cents.

A survey of the techniques for measuring the radio refractive index, R. E. McGavin, NBS Tech. Note 99 (May 1962), 30 cents.

Required signal-to-noise ratios, RF signal power, and bandwidth for multichannel radio communications systems, E. F. Florman and J. J. Tary, NBS Tech. Note 100 (Jan. 1962), $1.00.

A tabulation of the thermodynamic properties of normal hydrogen from low temperatures to 540° R from 10 to 1500 psia, Supplement A (British units), J. W. Dean, NBS Tech. Note 120A (June 1962) 45 cents.

Bibliography on direction finding and related ionospheric propagation topics, 1955–1961, O. D. Remmler, NBS Tech. Note 127, (Oct. 1962) 60 cents.

Controlled temperature oil baths for saturated standard cells, P. H. Lowrie, Jr., NBS Tech. Note 141, (Aug. 1962) 25 cents.

Equatorial spread F, W. Calvert, NBS Tech. Note 145 (Aug. 1, 1962) 60 cents.

General characteristics of linear strain gage accelerometers used in telemetry, P. S. Lederer, NBS Tech. Note 150 (June 1962) 40 cents.

Mode conversion in the earth-ionosphere waveguide, J. R. Wait, NBS Tech. Note 151 (June 8, 1962) 20 cents.

A general survey of the semiconductor field, G. W. Reimherr, NBS Tech. Note 153 (Aug. 1962) 30 cents.

The thermodynamic properties of helium from 3 to 300° K between 0.5 and 100 atmospheres, D. B. Mann, NBS Tech. Note 154 (Jan. 1962) 50 cents.

The energy parameter B for strong blast waves, D. L. Jones, NBS Tech. Note 155, (July 1962) 25 cents.

Information selection systems retrieving replica copies: A state-of-the-art report, T. C. Bagg and M. E. Stevens, NBS Tech. Note 157 (Dec. 31, 1961) $1.25.

Efficient use of the radio spectrum, K. A. Norton, NBS Tech. Note 158 (Apr. 1962) 45 cents.

A Fortran code for calculation of eigenvalues and eigenfunctions in real potential wells, R. S. Caswell, NBS Tech. Note 159 (Aug. 1962) 25 cents.

Evaluation of unexpectedly large radiation exposures by means of photographic film, W. L. McLaughlin, NBS Tech. Note 161, (Aug. 1962) 15 cents.

Photosensitized reaction between hydrogen (2p) rho atoms and molecular nitrogen, I. Tanaka and J. R. McNesby, J. Chem. Phys. **36,** No. 12, 3170–3173 (June 15, 1962).

Effect of particle size on low-temperature heat capacities, A. C. Victor, J. Chem. Phys. **36,** No. 10, 2812–2813 (May 15, 1962).

Physical standards of emittance and reflectance, J. C. Richmond, (Proc. Conf. Radiative Transfer from Solid Materials, Boston, Mass., Dec. 12–13, 1960), Book, Radiative Transfer from Solid Materials, ed. H. H. Blau, Jr., and H. Fischer, Sec. III, pp. 142–153 (The Macmillan Co., New York, N.Y., 1962).

Effect of molecular oxygen on the emission spectra of atomic oxygen-acetylene flames, S. L. N. G. Krishnamachari and H. P. Broida, J. Chem. Phys. **34,** 1709–1711 (1961).

Signs of nuclear resonance coupling constants in saturated aliphatic systems, H. Finegold, Proc. Chem. Soc. pp. 213–214 (June 1962).

The moropholy of mid-latitude 6300 angstrom arcs, T. Tohmatsu and F. E. Roach, J. Geophys. Res. **67,** No. 5, 1817–1821 (May 1962).

Some electrical properties of the porous graphite contact on *p*-type silicon, G. G. Harman, T. H. Higier, and O. L. Meyer, J. Appl. Phys. **33,** 2206 (July 1962).

The structure of the vibrational-rotational bands of an asymmetric rotor, H. C. Allen, Jr., Phil. Trans. Roy. Soc. London, Ser. B. Math. and Phys. Sci. **253,** No. 1030, 335–357 (Apr. 27, 1961).

A combined analog-digital differential analyzer (CADDA), W. D. Urban, W. R. Hahn, Jr., and H. K. Skramstad, Proc. Combined Analog Digital Computer Systems Symp., Phila., Pa., Dec. 1960, 2d item (1960).

The types of blackout, their time variations, and the mechanisms producing them, V. Agy, J. Phys. Soc. Japan **17,** 93–97 (Sept. 1961).

Some properties of dirty contacts on semi-conductors and resistivity measurements by a two terminal method, G. G. Harman and T. Higier, J. Appl. Phys. **33,** 2198 (July 1962).

A property of linear frequency modulation, A. J. Goldman, Proc. IRE **50,** No. 7, 1711 (July 1962).

The vapor pressure of 20 °K equilibrium hydrogen, L. C Weber, Jr., D. E. Diller, H. M. Roder, and R. D. Goodwin, Cyrogenic **3,** 236–238 (June 1962).

The thermal properties of powder insulators in the temperature range 300°—4° K, D. Cline and R. H. Kropschot (Proc. Conf. Radiative Transfer from Solid Materials, Boston, Mass., Dec. 12–13, 1960), Book, Radiative Transfer from Solid Materials, ed. H. H. Blau, Jr., and H. Fischer, sec. I, pp. 61–81 (The Macmillan Co., New York, N.Y., 1962).

Kinetics of the hydrolysis of acetal in *N*-methylpropionamide-water and *N*, *n*-dimethylformamide-water solvents at 20, 25, 30, and 40°, R. K. Wolford and R. G. Bates, J. Phys. Chem. **66,** No. 8, 1496–1500 (1962).

Symposium on spectroscopic excitation, B. F. Scribner, Am Soc. Testing Materials Spec. Tech. Publ. **259,** 1 (Oct. 1960).

Path loss measurements versus prediction for long distance tropospheric scatter circuits, A. F. Barghausen and C. F.

Peterson, IRE Trans. Commun. Systems **CS–9,** No. 4, 439–445 (Dec. 1961).

Introduction to the theory of V.L.F. propagation, J. R. Wait, Proc. IRE **50,** 1624–1647 (July 1962).

Some experimental aspects of nuclear orientation, E. Ambler, Proc. Tenth Intern. Congress of Refrigeration, Copenhagen, Denmark, I, 195–198 (1959)

The densities of saturated liquid hydrogen, R. D. Goodwin, D. E. Diller, H. M. Roder, and L. A. Weber, Cryogenics **2,** 81–83 (Dec. 1961).

Reference tables for 40% iridium-60% rhodium versus iridium thermocouples, G. F. Blackburn and F. R. Caldwell, Book, Temperature, Its Measurement and Control in Science and Industry **3,** Pt. 2, 161–175 (Reinhold Publ. Corp., New York, N.Y., 1962).

The cement reference laboratory (1929–1959), J. R. Dise, Am. Soc. Testing Materials Proc. 59, 369 (1959).

International coordination of measurement, A. G. McNish, Sci. Math. Weekly 2, No. 3, 28–29 & 35 (Sept. 1961).

The calibration at the National Bureau of Standards of mass standards for ultramicroanalysis, L. B. Macurdy, Book, Vacuum Microbalance Techniques, ed. R. F. Walker, 2, 165–175 (Plenum Press, Inc., New York, N. Y., 1962).

Post office mechanization, B. M. Levin, M. C. Stark and P. C. Tosini, Elec. Eng. 80, No. 2, 105–110 (Feb. 1961).

Switching properties in ferroelectrics of the family $Bi_4Ba_{m-2}Ti_{m+1}O_{3(m+2)}$, P. H. Fang and E. Fatuzzo, J. Phys. Soc. Japan **17,** 238 (1962).

Rate of vaporization of refractory substances, J. J. Diamond, J. Efimenko, R. F. Hampson, and R. F. Walker (Proc. 4th Intern. Symp. Reactivity of Solids), Book, Reactivity of Solids, ed. J. H. de Bower, et al., p. 725 (Elsevier Publ. Co., Amsterdam, The Netherlands, 1961).

A derivation of the relaxation spectrum representation of the mechanical response function, R. S. Marvin, Repts. Progr. Polymer Phys. Japan **5,** 56–58 (1962).

Thermal isomerization of isopropyl-1,1,1-d_3 radicals, W. M. Jackson and J. R. McNesby, J. Chem. Phys. **36,** No. 9, 2272–2275 (May 1, 1962).

Research on light metals in the Metallurgy Division, National Bureau of Standards, T. G. Digges, (Symp. Light Metal Industry, Jamshedpur, India) Indian Construction News, p. 68 (Aug. 1961).

A nomograph for selecting light balancing filters for camera exposure of color films, C. S. McCamy, Med Biol. Illustration, London, England **11,** No. 1, 13–15 (Jan. 1961).

Young's modulus of single crystal corundum from 77°K to 850 °K, J. B. Wachtman, Jr., W. E. Tefft, and D. G. Lam, Jr., Book, Mechanical Properties of Engineering Ceramics, pp. 221–223 (Interscience Publ. Inc., New York, N. Y., 1961).

Attempts to eliminate fatigue damage by heat treatment, J. A. Bennett, Am. Soc. Metals, Trans. Quart. (Tech. Notes) 55, No. 2, 362–363 (June 1962).

A controller for maintaining a constant rate of vaporization in fractional distillation, E. C. Kuehner and R. T. Leslie, Anal. Chem. **34,** No. 9, 1155–1156 (Aug. 1962).

Nuclear moment of NI^{61}, L. H. Bennett and R. L. Streever, Jr., Phys. Rev. Letters **121,** No. 6, 2141–2142 (June 15, 1962).

A method for determining mechanical resonance frequencies and for calculating elastic moduli from these frequencies, S. Spinner and W. E. Tefft, Am. Soc. Testing Materials Proc. **61,** 1221–1238 (1961).

New uses of microfilm with electronic scanners, a progress report on FOSDIC III, M. L. Greenough, Proc. 8th Annual Meeting Natl. Microfilm Assoc., Wash., D.C., Apr. 2–4, 1959, p. 278 (Annapolis, Md., 1959).

Susceptor elements for high temperature induction heating, S. Hasko and H. S. Parker, Am. Ceram. Soc. Bull. **41,** No. 7, 467 (July 1962).

The dynamic compressibility of a rubber-sulfur vulcanizate and its relation to free volume, J. E. McKinney, H. V. Belcher, and R. S. Marvin, Trans Soc. Rheology **4,** 347–362 (1960).

Relation of solar active regions at central meridian passage to ionospheric disturbance, C. S. Warwick and J. V. Lincoln, AGARDograph Proc. Sixth AGARD Ionospheric Research Committee Meeting (Rome, Italy, May 15–18, 1961).

Suggested arrangement of mirrors to form multiple reference angles, J. B. Saunders, J. Opt. Soc. Am. **51,** No. 8, 859–862 (Aug. 1961).

Radiolysis of ethane-1,1,1-d_3, L. J. Stief and P. Ausloos, J. Chem. Phys. **36,** No. 11, 2904 (June 1, 1962).

D–C differential current meter, E. Niesen, Rev. Sci. Instr. **32,** No. 12, 1407–1408 (Dec. 1961).

Radiowave propagation during World War II, K. L. Norton, Proc. IRE **50,** No. 5, 698–704 (May 1962).

Development directions to keep leather apace, J. R. Kanagy, Leather and Shoes **142,** 34–35 (Aug. 12, 1961).

High-energy X-ray spectrometer using large anticoincidence sodium iodide crystals, J. M. Wycoff, Proc. Total Absorption Gamma-Ray Spectrometry Symp., Gatlinburg, Tenn., May 10, 1960, pp. 201–210 (1960).

Radio propagation quality in the North Atlantic area 1953–1960, M. E. Nason, AGARDograph Proc. 6th AGARD Ionospheric Research Committee Meeting (Rome, Italy, May 15–18, 1961).

Discussion of Feldman's and Sereda's Paper "Characteristics of sorption and expansion isotherms of reactive limestone aggregate," C. M. Hunt, L. A. Tomes, R. G. Pike, and D. Hubbard, J. Am. Concrete Inst. **59,** 815–818 (Mar. 1962).

The National Bureau of Standards Library, S. A. Jones, D.C. Libraries **32,** 7–10 (Jan. 1961).

Systematic errors, W. J. Youden, Ordnance **XLVI,** No. 248, 299–301 (Sept.–Oct. 1961).

Determination of the dissociation equilibria of water by a conductance method, H. C. Duecker and W. Haller, J. Phys. Chem. **66,** No. 2, 225–229 (Feb. 1962).

Heat of formation of the most stable form of metaboric acid, HBO_2(cI), M. V. Kilday and E. J. Prosen, J. Am. Chem. Soc. Commun. to Editor **82,** 550 (1960).

The scientific contributions of William Weber Coblentz, E. K. Plyler, J. Appl. Spect. **16,** No. 3, 73–77 (1962).

Absorption spectrum of CF_2 trapped in an argon matrix, A. M. Bass and D. E. Mann, J. Chem. Phys. **36,** No. 12, 3501–3502 (June 15, 1962).

The rapid selector and other NBS document retrieval studies, J. L. Pike and T. C. Bagg, Proc. National Microfilm Assoc., Annual Meeting, Wash., D.C., Apr. 25–27, 1962, XI, 213–227 (Annapolis, Md., 1962).

A test method for air-entrainment of standard Ottawa sand, M. R. DeFore and H. J. Corah, ASTM Bull. No. 248, 48–56 (TP162–166) (Sept. 1960).

The angular distribution of beta particles from oriented cerium-141 nuclei, D. D. Hoppes, Proc. Seventh Intern. Conf. Low Temperature Physics, University of Toronto, Canada, Aug. 1960, p. 186 (University of Toronto Press, Toronto, Canada, 1960).

Die radiometeorologie and ihre bedeutung fur die ausbreitung der m-, dm- and cm- wallen auf grosse entfernungen, B. R. Bean, L. Fehlhaber, and J. Grosskopf, Nachr. Z (NTZ), p. 9–16 (Jan. 1962).

The program at the National Bureau of Standards, L. M. Branscomb, Book, Optical Spectroscopic Measurements of High Temperatures, ed. P. J. Dickerman, p. 235 (Univ. of Chicago Press, Chicago, Ill. 1961).

Determination of oxidation rates of air-blown asphalts by infrared spectroscopy, J. R. Wright and P. G. Campbell, J. Appl. Chem. **12,** 256–266 (June 1962).

Thermocouple materials, F. R. Caldwell, Temp. Meas. Control. Sci. Ind. **3,** pt. 2, 81–134 (Reinhold Publ., Corp., New York, N.Y., 1962).

Study of "valley problem" with a ray tracing program, K. Davies and A. K. Saha, Electron Density Profiles, pp. 162–166 (1962).

The extrapolation of the orthorhombic *N*-paraffin melting properties to very long chain lengths, M. G. Broadhurst, J. Chem. Phys. **36,** No. 10, 2578 (May 15, 1962).

An equation of state for calculating the thermodynamic properties of helium at low temperatures, R. D. McCarty and R. B. Stewart, Book, Progress in International Research on Thermodynamic and Transport Properties, pp. 107–117 (Academic Press, Inc., New York, N.Y., 1962).

Thermal expansion of some engineering materials from 20° K to 293° K, V. Arp, J. H. Wilson, L. Winrich, and P. Sikora, Cryogenics **3,** 230 (June 1962).

Interlaboratory evaluation of a method for indicated bright-

ness of papers containing fluorescent brighteners, T. W. Lashof and J. M. Patek, Tappi **45,** No. 7, 566 (July 1962).

An intermediate size automatically controlled hydrogen refrigeration system, D. B. Chelton, D. B. Mann, and B.W. Birmingham, Suppl. Bull. Inst. Intern du Froid, Eindhoven, Comm. 1, Annexe 1960–61, 169–178 (1960).

Thermal radiation standards and measurements of the Radiometry Section at the National Bureau of Standards, A. G. Maki, (Proc. Conf. Radiative Transfer From Solid Materials, Boston, Mass., Dec. 12–13, 1960), Book, Radiative Transfer from Solid Materials, ed. H. H. Blau, Jr., and H. Fischer, Sec. III, pp. 135–141 (The Macmillan Co., New York, N. Y., 1962).

Absolute microwave refractometer, M. J. Vetter and M. C. Thompson, Jr., Rev. Sci. Instr. **33,** 656–660 (June 1962).

The information problem in government, S. N. Alexander, Proc. Engineering Information Symp. (Sponsored by the Engineers Joint Council), pp. 15–16 (Jan. 17, 1962).

Problems in the temperature calibration of an X-ray diffractometer furnace, F. A. Mauer and L. H. Bolz, (Proc. Tenth Annual Conf. Applications of X-ray Analysis, Denver, Colo., Aug. 7–9, 1961), Book, Advances in X-ray analysis, p. 229 (Plenum Press, Inc., New York, N.Y., 1962).

The detection and study of solar cosmic rays by radio techniques, D. K. Bailey, (Intern. Conf. on Cosmic Rays and the Earth Storm), J. Phys. Soc. Japan **17,** Suppl. A–I, pt. I, 106–112 (1962).

Precision and accuracy—experiment design aspects, C. Eisenhart, Conf. on Applications of Statistical Methods in the Chemical Industry, Feb. 4, 1961, (Sponsored by Am. Soc. for Quality Control, Metropolitan Section, Seaton Hall University, South Orange, N.J.), p. 1 (1961).

Purification and vapor pressure of pure nitric oxide, E. E. Hughes, J. Chem. Phys. Letter to Editor **35,** No. 4, 1531–1532 (Oct. 1961).

Ultrasounds induce flaking of ceramics from metals, J. W. Pitts, Metal Progr. **82,** No. 2, 114–115 (Aug. 1962).

Ferroelectricity in the compound $Ba_2Bi_4Ti_5O_{18}$, P. H. Fang and B. Aurivillius, Phys. Rev. **126,** No. 3, 893 (May 1962).

The ammonia beam maser as a standard of frequency, J. A. Barnes, D. W. Allan, and A. E. Wainwright, IRE Trans. Instrumentation **I–11,** 26–30 (June 1962).

Use of gas phase chromatography for rapid determination of carbonate at low levels, F. G. Carpenter, Anal. Chem. **34,** 66 (Jan. 1962).

An experimental investigation of the scintillation of radio stars observed at frequencies of 223 and 456 megacycles per second from a location close to the auroral zone, C. G. Little, G. C. Reid, E. Stiltner, and R. P. Merritt, J Geophys. Res. **67,** No. 5, 1763–1784 (May 1962).

The radiative formation and destruction of negative ions, L. M. Branscomb, Proc. Fifth Intern. Conf. Ionization Phenomena in Gases, Munich, Germany, I, 1–18 (North Holland Publ. Co., Amsterdam, The Netherlands, 1961).

A computer for weather data acquisition, P. Meissner, J. A. Cunningham, and C. A. Kettering, Proc. Eastern Joint Computer Conf., Dec. 13–15, 1960, p. 57 (New York, N.Y., 1960).

Position titles of chemists, W. K. Wilson, Capital Chemist 10, 145 (May 1960).

Corrected calculations of sound absorption in non-associated liquids, R. E. Nettleton, J. Acoust. Soc. Am. **34,** No. 3, 350 (Mar. 1962).

International comparisons of radioactivity standards, W. E., Perry and W. B. Mann, Proc, Ninth Intern. Congress on Radiology, Munich, Germany, July 1959, 23, VII–30. VII, 1338–1342 (Urban and Schwarzenberg, Munich, Germany, 1960).

A rapid simple method of estimating the order of chemical reactions, J. H. Flynn, Chem. Eng. **69,** No. 17, 137–104 (Aug. 20, 1962).

Paramagnetic resonance phenomena, H. E. Radford, Encyclopaedic Dictionary of Physics **5,** 293 (Pergamon Press Inc., New York, N.Y., 1962).

Chilled-air-distribution in refrigerated trailers, P. R. Achenbach, Suppl. Bull. Intern. Inst. of Refrigeration, p. 9 (1961–1962).

Mass spectrum and appearance potentials of tetrafluorohydrazine, J. T. Herron and V. H. Dibeler, J. Chem. Phys. **33,** No. 5. 1595–1596 (Nov. 1960).

Large longitudinal retarded elastic deformation of rubberlike network polymers, H. Leaderman, J. Polymer Sci. **59,** No. 168, S42 (June 1962).

A survey of polar cap absorption events (solar proton events) in the period 1952 through 1960, D. K. Bailey and J. M. Harrington, (Intern. Conf. on Cosmic Rays and the Earth Storm), J. Phys. Soc. Japan **17,** Supply. A–II, pt. II, 334–337 (1962).

Magnifications of a telescope, R. E. Stephens, J. Opt. Soc. Am. **51,** No. 7, 803–804 (July 1961).

Isotope effect in an electric field and jump frequencies for diffusion in ionic crystals, J. R. Manning, J. Appl. Phys. **33,** No. 7, 2145–2151 (July 1962).

Determination of isotope effects by "double labeling" oxidation of *d*-glucose with iodine, H. S. Isbell, L. T. Sniegoski, and H. L. Frush, Anal. Chem. **34,** No. 8, 982–984 (July 1962).

Methods of nuclear orientation, E. Ambler, Book, Progress in Cryogenics, ed. K. Mendelssohn III, 235–280 (Heywood and Co., Ltd., London, England, 1960).

Linear thermal expansion of aluminum oxide and thorium oxide from 100° to 1100° K., J. B. Wachtman, Jr., T. G. Scuderi, and G. W. Cleek, J. Am. Ceram. Soc. **45,** 310–323 (July 1962).

Molecular collision models and transition probabilities in liquids, R. E. Nettleton, J. Chem. Phys. **38,** 2226–2227 (Apr. 1962).

Electronic development and production in the USSR, C. P. Marsden, Encylcopedia on Russia and the Soviet Union, pp. 158–160 (McGraw-Hill Book Co., Inc., New York, N.Y., 1961).

Microwave spectrum of trimethylarsine, D. R. Lide, Jr., Spectrochem. Acta 15, 473 (1959).

Lattice frequencies and rotational barriers for inorganic carbonates and nitrates, R. A. Schroeder, C. E. Weir, and E. R. Lippincott, J. Chem. Phys. **36,** No. 10, 2803–2804 (May 1962).

Ferroelectricity in the compound $Bi_4Ti_3O_{12}$, P. H. Fang and C. R. Robbins, Phys. Rev. **126,** No. 3, 892 (May 1962).

Environmental characteristics of a small underground fallout shelter, P. R. Achenbach, F. J. J. Drapeau, and C. W. Phillips, ASHRAE J. **4,** No. 1, 21 (Jan. 1962).

Opportunities in dental research, G. C. Paffenbarger, J. Am. Dental Assoc. **60,** No. 3, 413–414 (Mar. 1960).

On the nature of equatorial slant sporadic *E*, R. Cohen, K. L. Bowles and W. Calvert, J. Geophys. Res. **67,** No. 3, 965–972 (Mar. 1962).

Elementary statistical design, W. J. Youden, Conf. on Applications of Statistical Methods in the Chemical Industry, Feb. 4, 1961, (Sponsored by Am. Soc. for Quality Control, Metropolitan Section, Seaton Hall University, South Orange, N.J.), p. 43 (1961).

Dielectric loss in "non-polar" polymers, A. J. Curtis, J. Chem. Phys. **36,** No. 12, 3500 (June 15, 1962).

Mobilities of positive ions in argon, E. C. Beaty, Proc, Conf. Ionization Phenomena in Gases, Aug. 28–Sept. 1. 1961, Munich, Germany, p. 183 (North Holland Publ. Col., Amsterdam, The Netherlands, 1962).

Blue-glass filters to approximate the blackbody at 6500° K, D. B. Judd, Farbe **10,** No. 1–4, 31 (1961).

Correction to "doppler studies of the ionosphere with vertical incidence," K. Davies, Proc. IRE **50,** No. 6, 94 (June 1962).

Electron photodetachment from ions and elastic collision cross sections for O, C, Cl and F, J. W. Cooper and J. B. Martin, Phys. Rev. **126,** No. 4, 1482 (May 15, 1962).

The following articles were presented before the 1961 Cryogenic Engineering Conference and published in the Book, Advances in Cryogenic Engineering **7** (Plenum Press, Inc., New York, N.Y., 1962):

A cryostat for tensile tests in the temperature range 300° to 4° K, R. P. Reed, Paper K–3, 448–454.

A method of measuring shear modulus from −424° to 70° F, R. P. Mikesell and R. M. McClintock, Paper K–11509–513.

An electrically controlled guarded flat plate calorimeter, D. Cline and R. H. Kropschot, Paper L–5, 534–538.

An experimental investigation of the individual boiling and

condensing heat transfer coefficients for hydrogen, D. E. Drayer and K. D. Timmerhaus, Paper J–6, 401–412.

Dry gas operation of ball bearings at cryogenic temperatures, L. E. Scott, D. B. Chelton, and J. A. Brennan, Paper G–3, 273–276.

Low temperature characteristics of some commercial thermocouples, R. L. Powell and L. P. Caywood, Jr., Paper L–2, 517–521.

Martensitic transformation products and mechanical properties of austenitic stainless steels at low temperatures, C. J. Guntner and R. P. Reed, Paper K–9, 500–502.

On the bulk density of boiling liquid oxygen, R. W. Arnett, Paper F–4, 214–218.

Temperature dependence of magnetic losses, J. J. Gniewek and R. L. Powell, Paper H–1, 303–310.

Testing of ball bearings with five different separator materials at 9200 RPM in liquid nitrogen, J. A. Brennan, W. A. Wilson, R. Radebaugh, and B. W. Birmingham, Paper G–2, 262–272.

The application and behavior of elastomers at cryogenic temperatures, R. F. Robbins, D. H. Weitzel, and R. N. Herring, Paper H–6, 343–352.

The application of gas-lubricated bearings to a miniature helium expansion turbine, B. W. Birmingham, H. Sixsmith, and W. A. Wilson, Paper A–4, 30–42.

The efficiency of an ideal refrigerator, R. B. Jacobs, Paper L–11, 567–571.

Application of reflectometer techniques to accurate reflection measurements in coaxial systems, R. W. Beatty and W. J. Anson, Proc. Inst. Elec. Engrs. **B109,** 345–348 (July 1962).

Chemically induced vibrational excitation: Hydroxyl radical emission in the 1–3-micron region produced by the $H + O_3$ atomic flame, D. Garvin, H. P. Broida, and H. J. Kostkowski, J. Chem. Phys. **37,** No. 1, 193 (July 1962).

Gas-phase radiolysis of propane, P. Ausloos and S. Lias, J. Chem. Phys. **36,** No. 12, 3163–3170 (July 15, 1962).

Four methods for predicting the durability of roofing asphalts, S. H. Greenfeld and J. R. Wright, Mater. Res. Std. **2,** No. 9, 738–745 (Sept. 1962).

The absolute simplest form in digital circuit design, R. S. Ledley, Book, Digital Computers and Control Engineering, 835 pgs., (McGraw-Hill Book Co., Inc., New York, N.Y., 1960).

Infrared spectrum of dideuteroacetylene (C_2D_2), E. D. Tidwell and E. K. Plyler, J. Opt. Soc. Am. **52,** No. 6, 656–664 (June 1962).

Coring an electroform, W. H. Metzer, Jr., Plating 49, No. **8,** 880 (Aug. 1962).

The standards challenge, A. H. Scott, Insulation **8,** No. 2, 48–50 (Feb. 1962).

A system for generating "pronounceable" names using a computer, A. L. Leiner and W. W. Youden, J. Assoc. Computing Mach. **8,** No. 1, 97–103 (Jan. 1961).

Using digital computers in the design and maintenance of new computers, A. L. Leiner, A. Weinberger, H. Loberman, and C. Coleman, IRE Trans. Electronic Computers **EC–10,** No. 4, 680–690 (Dec. 1961).

Surface preparation of solid metallic samples for X-ray spectrochemical analysis, R. E. Michaelis and B. A. Kilday, Book, Advances in X-ray Analysis **5,** 405–411 (Plenum Press, Inc., New York, N.Y., 1962).

Infrared wavelength dependence of the total absorptivity of electroplated silver, D. Cline, J. Appl. Phys. **33,** 2310–2311. (July 1962).

Simple stain gauge-based load controller, L. Mordfin and R. L. Bloss, Rev. Sci. Instr. **33,** No. 7, 772–773 (July 1962).

Quantum-mechanical calculation of harmonic oscillator transition probabilities. II. Three-dimensional impulsive collisions, F. H. Mies and K. E. Shuler, J. Chem. Phys. **37,** No. 1, 177–181 (July 1, 1962).

Remote phase control of station WWV, R. F. Fey, J. B. Milton, and A. H. Morgan, Nature **193,** No. 4820, 1063–1064 (Mar. 17, 1962).

Melting and contractility of feather keratin, L. Mandelkern, J. C. Halpin, and A. F. Diorio, J. Polymer Sci. Letter to Editor **60,** No. 169, S31–S33 (July 1962).

Wave propagation around a curved boundary which contains an obstacle, J. R. Wait, Can. J. Phys. **40,** 1010–1016 (Aug. 1962).

Four-place table decibels return loss to magnitude of voltage reflection coefficient, R. W. Beatty, Microwave Eng. Handb. & Buyers Guide, pp. TD188–192 (Nov. 1961).

Duration and spacing of sferic pulses, R. F. Linfield and C. A. Samson, Proc. IRE **50,** 1841–1842 (Aug. 1962).

VHF and UHF signal characteristics observed on a long knife-edge, A. P. Barsis and R. S. Kirby, IRE Natl. Conv. Record, pp. 17–34 (1961).

Some observations on growing crystals of argon, L. H. Bolz, H. P. Broida, and H. S. Peiser, Acta Cryst. **15,** No. 8, 810–812 (Aug. 1962).

Methods and techniques of low and very low frequency monitoring at Boulder Laboratories, A. H. Morgan and D. H. Andrews, Consultative Committee for Definition of the Second Intern. Comm. on Weights and Measures Apr. 11–12, 1961, pp. 1–10 (June 1961).

Insulation resistance measurements, A. H. Scott, Proc. 4th Electrical Insulation Conf., Materials and Application, Wash., D.C., p. 115–117 (Feb. 19–22, 1962).

Physical measurement: Pilot of progress, A. V. Astin, Steel, p. 112 (July 23, 1962).

Some tensile properties of amalgam, M. S. Rodriguez and G. Dickson, J. Dental Res. **41,** No. 4, 840–852 (July–Aug. 1962).

Vacuum photolysis of solid ethane at 77° K, M. D. Scheer, J. McNesby, and R. Klein, J. Chem. Phys. **36,** No. 12, 3504–3505 (June 1962).

Geomagnetic and solar data, J. V. Lincoln, J. Geophys. Res. **67,** No. 5, 2035–2038 (May 1962).

A study of the chemical and physical properties of magnetic recording tape, F. Nesh and R. F. Brown, Jr., IRE Trans. Audio **AU–10,** No. 3, 70–71 (May–June 1962).

Supporting evidence for solar flare effects in the *F* region of the ionosphere, R. W. Knecht and K. Davies, Nature **192,** No. 4800, 348–350 (Oct. 28, 1961).

Measurement of effect of moisture on heat transfer through insulated flat-roof constructions, F. J. Powell and H. E. Robinson, Am. Soc. Testing Materials Spec. Tech. Publ. **312,** 35–66 (1961).

Dependence of interval between flare and associated sudden commencement storm on prestorm conditions, M. W. Haurwitz, J. Geophys. Res. Letter **67,** 2979–2982 (July 1962).

The application of NMR in determination of the structure of cyclanols. I. The structures of cyclohexane-1,3-diols, H. Finegold and H. Kwart, J. Organic Chem. **27,** 2361–2365 (1962).

APPA–TAPPI reference material program. I. Interlaboratory investigation of TAPI standard T 414 m–49, internal tearing resistance of paper, T. W. Lashof, Tappi **45,** No. 8, 656 (1962).

Spectrophotofluorometric studies of degraded cotton cellulose, S. D. Toner and K. F. Plitt, Tappi **45,** No. 8, 681–688 (Aug. 1962).

Properties of dental amalgams made from spherical alloy particles, N. C. Demaree and D. F. Taylor, J. Dental Res. **41,** No. 4, 890–906 (July–Aug. 1962).

Emission spectra of solids condensed at very low temperatures from the electrical discharge products of nitrogen and carbon monoxide or acetylene, S. L. N. G. Krishnamachari, R. W. Nicholls, and H. P. Broida, Proc. Indian Acad. Sci. **LIV,** 61–68 (1961).

Excitation of the red and green coronal lines, C. Pecker and R. N. Thomas, Ann. Astrophys. **25,** 100–108 (Mar.–Apr. 1962).

Dependence of the ionospheric *F* region on the solar cycle, J. W. Wright, Nature **194,** No. 4827, 461–462 (May 1962).

Fast-melting alloy forms water jacket for small klystrons, W. J. Anson and E. Niesen, Electron. Design, pp. 42–45 (Mar. 1962).

Boolean matrix equations in digital circuit design, R. S. Ledley, IRE Trans. Electronic Computers **EC–8,** No. 2, 131–139 (June 1959).

Spectrophotometric determination of hydroperoxide in diethyle ether, W. C. Wolfe, Anal. Chem. **34,** No. 10, 1328–1330 (Sept. 1962).

Exchange behavior of kaolins of varying degrees of crystallinity, W. C. Ormsby, J. M. Shartsis and K. H. Woodside, J. Am. Ceram. Soc. **45,** No. 8, 361–366 (Aug. 1962).

Analysis of the hydroxyl radical vibration rotation spectrum between 3900 Å and 15000 A, A. M. Bass and D. Garvin, J. Mol. Spectry. **9,** No. 2, 114–123 (Aug. 1962).

Microwave measurements in the NBS electronic calibration center, R. E. Larson, Inst. Elec. Engrs. **109,** pt. B, Suppl. No. 23, 644–650 (1962).

Structure and structure imperfections of solid β-oxygen, E. M. Horl, Acta. Cryst. **15,** No. 9, 845–850 (Sept. 1962).

Kinetics of Cs^+ desorption from tungsten, M. D. Scheer and J. Fine, J. Chem. Phys. **37,** No. 1, 107–113 (July 1962).

Effect of additives on silver iodide particles exposed to light, G. Burley and D. W. Herrin, J. Appl. Meteorol. **1,** No. 3, 355–356 (Oct. 1962).

A modulated sub-carrier technique of measuring microwave attenuation, G. E. Schafer and R. R. Bowman, Inst. Elec. Engrs. **109,** pt B, Suppl. 23, 783–786 (1962).

A survey of microwave power-measurement techniques employed at the National Bureau of Standards, G. F. Engen, Inst. Elec. Engrs. **109,** pt. B, Suppl. No. 23, 734–739 (1962).

Symposium on teaching of materials: Aspects of material behavior significant to engineers, J. M. Frankland, J. Eng. Mech. Div.—Proc. Am. Soc. Civil Engr. Paper No. 3213, 75–81 (Aug. 1962).

Neutral meson photoproduction from complex nuclei, R. A. Schrack, J. E. Leiss and S. Penner, Phys. Rev. **127,** No. 5, 1772–1783 (Sept. 1, 1962).

Excitation of modes at very low frequency in the earth-ionosphere wave guide, J. R. Wait, J. Geophys. Res. **67,** No. 10, 3823–3828 (Sept. 1962).

The growth and roots of electronic computers, J. L. Little, Student Quart. and EE Digest, Joint Publ. of Am. Inst. Elec. Engr. and IRE **1,** No. 1, 64–70 (Sept. 1962).

Acoustical interferometer employed as an instrument for measuring low absolute temperatures, G. Cataland and H. H. Plumb, J. Acoust, Soc. Am. **34,** No. 8, 1145–1146 (Aug. 1962).

**Publications for which a price is indicated are available from the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. (foreign postage one-fourth additional). Reprints from outside journals and the NBS Journal of Research may often be obtained directly from the authors.*

U.S. GOVERNMENT PRINTING OFFICE:1963

UNIVERSITY MICROFILMS INC
EDITORIAL DEPARTMENT
313 NORTH FIRST ST
ANN ARBOR MICH
S-IR-B JUNE

UNITED STATES
GOVERNMENT PRINTING OFFICE
DIVISION OF PUBLIC DOCUMENTS
WASHINGTON 25, D.C.

OFFICIAL BUSINESS

OR PRIVATE USE TO AVOID
PAYMENT OF POSTAGE, $300
(GPO)

JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 67B, No. 1 January–March 1963
B. Mathematics and Mathematical Physics

Contents

	Page
Evaluation of a generalized elliptic-type integral. Leo F. Epstein and J. H. Hubbell.	1
An algorithm for obtaining an orthogonal set of individual degrees of freedom for error. Joseph M. Cameron.	19
Recognition of completely mixed games. A. J. Goldman.	23
A new type of computable inductor. Chester H. Page.	31
Numerical computation of the temporal development of currents in a gas discharge tube. W. Börsch-Supan and H. Oser.	41
Tables of genera of groups of linear fractional transformations. Harriet Fell, Morris Newman, and Edward Ordman.	61
Publications of the National Bureau of Standards.	69

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. Price 75 cents (single copy). Subscription price: $2.25 a year $0.50 additional for foreign mailing.

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1963